The error in multivariate polynomial interpolation

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plenus venter / non studet libenter

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Multivariate polynomial interpolation

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Issue: $f - P_{\mathrm{T}} f \; = ???$

error bounds: an example

$\#\mathrm{T} = d + 1$ in general position, hence $F = \Pi_{\le 1}$ is standard choice.

M := Lipschitz constant of f. (Cooper'94) On conv(T),

$$|f - Pf| = |f - \sum_{\tau \in \mathrm{T}} \ell_\tau \, f(\tau)| \le \sum_{\tau \in \mathrm{T}} \ell_\tau \, |f - f(\tau)| \le M \, \mathrm{diam}(\mathrm{conv}(\mathrm{T}))$$

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Can do better:

$$|f(x) - Pf(x)| \le \sum_{\tau \in \mathrm{T}} |\ell_\tau(x)| \, |f(x) - f(\tau)| \le M \underbrace{\sum_{\tau \in \mathrm{T}} |\ell_\tau(x)| \, \|x - \tau\|}_{=: \varphi(x)}$$

dB'94: $\varphi(x) \le r$ whenever $x \in \mathrm{conv}(\mathrm{T}) \subset B_r(c)$ (e.g., circumscribed ball).

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Wlog $c = 0$. For any $x \in B_r(0) \supset \mathrm{conv}(\mathrm{T})$, $\quad b : y \mapsto \sqrt{r^2 - 2\langle x, y\rangle + \|x\|^2}$ is concave on $B_r(0)$, as the (positive) squareroot of the affine map

$$y \mapsto r^2 - 2\langle x, y\rangle + \|x\|^2 \;\ge\; \|x-y\|^2,$$

hence, for $x \in \mathrm{conv}(\mathrm{T})$,

$$\varphi(x) = \sum_{\tau\in\mathrm{T}} \ell_\tau(x)\|x-\tau\| \le \sum_{\tau} \ell_\tau(x) b(\tau) \le b(\sum_{\tau\in\mathrm{T}} \ell_\tau(x)\tau) = b(x) = \sqrt{r^2 - \|x\|^2}$$

Error in univariate polynomial interpolation

$$\begin{aligned}
f(x) - P_{\mathrm{T}} f(x) &= [\mathrm{T}, x] f \prod_{\tau \in \mathrm{T}} (x - \tau) \\
&= \int M(y|\mathrm{T}, x) D^{\#\mathrm{T}} f(y)\, \mathrm{d}y \prod_{\tau \in \mathrm{T}} (x - \tau) \\
&=: \int M(y|\mathrm{T}, x) q_{\uparrow}(D) f(y)\, \mathrm{d}y \;\; q(x) \\
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(Genocchi-Hermite) with

$$\int_{[t_1, \ldots, t_{n+1}]} g =: \int_0^1 \int_0^{s_1} \cdots \int_0^{s_{n-1}} g(t_1 + s_1 \Delta t_1 + \cdots + s_n \Delta t_n)\, \mathrm{d}s_n \cdots \mathrm{d}s_1$$

(with $\Delta t_i := t_{i+1} - t_i$) called by Micchelli the divided difference functional even when g is multivariate.

- The functional is symmetric in the t_j.
- The integral is over the set $\{t_1 + s_1 \Delta t_1 + \cdots + s_n \Delta t_n : 0 \le s_n \le \cdots \le s_1 \le 1\} = \mathrm{conv}\{t_1, \ldots, t_{n+1}\}$, i.e., the convex hull of $\{t_1, \ldots, t_{n+1}\}$ parametrized as a simplex.

multivariate divided difference?

MULTIvariate divided difference

Definition. *A kth divided difference is the composition of k first divided differences.*

first divided difference: $D_\xi g(x) := \lim_{t\to 0}(g(x+t\xi)-g(x))/t$

$$\begin{aligned}[x,y \mid \xi]g \; &:= \int_0^1 D_\xi g(x+s(y-x))\ \mathrm{d}s, \qquad x,y,\xi \in \mathbb{R}^d \\ &= \int_{[x,y]} D_\xi g\end{aligned}$$

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second divided difference:

$$\begin{aligned}[x,y,z\mid\xi,\eta]g \;&:=\; [y,z\mid\eta][x,\bullet\mid\xi]g\\ &= \int_0^1 D_\eta\Bigl(\int_0^1 D_\xi g(x+s_1(\bullet-x)\,\mathrm{d}s_1\Bigr)(y+s(z-y))\,\mathrm{d}s\\ &= \int_0^1\int_0^1 (D_\eta D_\xi g)(x+s_1((y+s(z-y))-x)s_1\,\mathrm{d}s\,\mathrm{d}s_1\\ &= \int_0^1\int_0^1 (D_\eta D_\xi g)(x+s_1(y-x)+\underbrace{s_1 s}_{:=s_2}(z-y))s_1\,\mathrm{d}s\,\mathrm{d}s_1\\ &= \int_{[x,y,z]} D_\eta D_\xi g\end{aligned}$$

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kth divided difference:

$$\begin{aligned}[x_1,\ldots,x_{k+1} \mid \xi_1,\ldots,\xi_k]g \;&:=\; [x_k,x_{k+1}\mid \xi_k][x_{k-1},\bullet \mid \xi_{k-1}]\cdots[x_1,\bullet\mid\xi_1]g\\ &= \int_{[x_1,\ldots,x_{k+1}]} D_{\xi_1}\cdots D_{\xi_k} g\end{aligned}$$

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symmetric in the *sites* x_i ; symmetric and multilinear in the *directions* ξ_j.

Micchelli'78: $g \mapsto [x_{\le k+1} \mid \xi_{\le k}]g$ is continuous on C iff $\xi_{\le k} \in \flat\{x_{\le k+1}\}$.

bootstrap

$$[x]g \quad = \quad [x_0]g \ + \ \underbrace{[x_0, x \mid x - x_0]g}_{[x_0, x_1 \mid x - x_0]g \ + \ [x_1, x \mid x - x_1][x_0, \bullet \mid x - x_0]g}$$

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$$
\begin{aligned}
[x]g &= [x_0]g + \underbrace{[x_0, x \mid x - x_0]g}_{} \\
&\qquad\qquad [x_0, x_1 \mid x - x_0]g + [x_1, x \mid x - x_1][x_0, \bullet \mid x - x_0]g \\
&=: \quad (P_1 g)(x) \qquad\qquad + \underbrace{[x_0, x_1, x \mid x - x_0, x - x_1]g}_{}
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$$[x_0, x_1, x_2 \mid x - x_0, x - x_1]g \; +$$

$$+ \; [x_2, x \mid x - x_2][x_0, x_1, \bullet \mid x - x_0, x - x_1]g$$

bootstrap

$$\begin{aligned}
[x]g \;&=\; [x_0]g \;+\; \underbrace{[x_0, x \mid x - x_0]g}_{[x_0, x_1 \mid x - x_0]g \;+\; [x_1, x \mid x - x_1][x_0, \bullet \mid x - x_0]g} \\
&=: \quad (P_1 g)(x) \quad + \; \underbrace{[x_0, x_1, x \mid x - x_0, x - x_1]g}_{\substack{[x_0, x_1, x_2 \mid x - x_0, x - x_1]g \;+ \\ +\; [x_2, x \mid x - x_2][x_0, x_1, \bullet \mid x - x_0, x - x_1]g}} \\
&=: \quad (P_2 g)(x) \quad + \; [x_0, x_1, x_2, x \mid x - x_0, x - x_1, x - x_2]g
\end{aligned}$$

Leads to Micchelli'78 formulation of **Kergin'78 interpolation**:

$$\begin{aligned}
[x]g \;=\; \Bigl((P_k g)(x) := \sum_{j=0}^{k} [x_0, \ldots, x_j \mid x - x_0, \ldots, x - x_{j-1}]g\Bigr) + \\
+\; [x_0, \ldots, x_k, x \mid x - x_0, \ldots, x - x_k]g
\end{aligned}$$

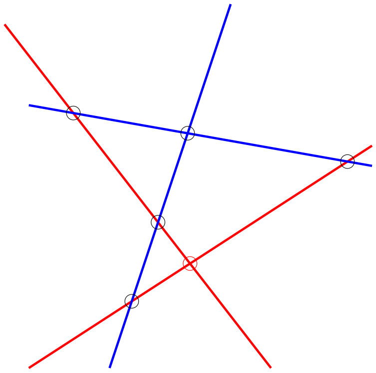
↓

- P_k is a linear projector onto $\Pi_{\le k}$, with
- $(\ker P_k)^\perp = \operatorname{span}\{p \mapsto \int_{[\Sigma]} q(D)p \;:\; q \in \Pi^0_{\#\Sigma - 1}, \Sigma \subset \mathrm{T}\}$.

Chung-Yao'77 interpolation

$\mathbb{H}$ a generic $d+k$-subset of $\Pi_1(\mathbb{R}^d)\backslash\Pi_0(\mathbb{R}^d)$

$$P_{\mathbb{H}}g := \sum_{H\in\binom{\mathbb{H}}{d}} g(\tau_H) \prod_{h\notin H} \frac{h}{h(\tau_H)} \qquad \in \Pi_{\le k}$$



matches g on

$$\mathrm{T} := \{\tau_H : H \in \binom{\mathbb{H}}{d}\}$$

with τ_H the unique site satisfying

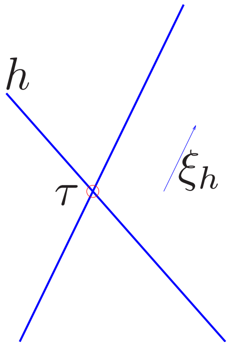
$$h(\tau_H) = 0 \quad \Longleftrightarrow \quad h \in H \ .$$

error for $k = 0$

$k = 0$ implies $\#\mathbb{H} = d$, so $\{\tau\} := \cap\mathbb{H}$, $h(x) = h_\uparrow(x-\tau) \ = \ D_{x-\tau}h$.

For $h \in \mathbb{H}$, choose ξ_h so that

$$h'_\uparrow(\xi_h) = 0 \quad \iff \quad h' \neq h.$$



Then $(\xi_h : h \in \mathbb{H})$ is a basis for $\mathbb{R}^d$, and

$$x - \tau = \sum_{h\in\mathbb{H}} \frac{h(x)}{h_\uparrow(\xi_h)}\xi_h,$$

therefore

$$g(x) - P_{\mathbb{H}}g(x) = g(x) - g(\tau) = [\tau, x \mid x-\tau]g = \sum_{h\in\mathbb{H}} \frac{h(x)}{h_\uparrow(\xi_h)}[\tau, x \mid \xi_h]g$$

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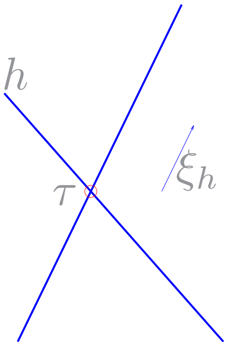

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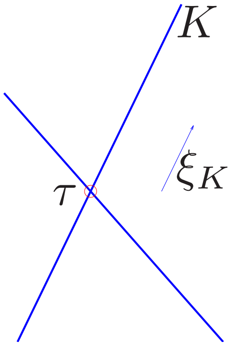
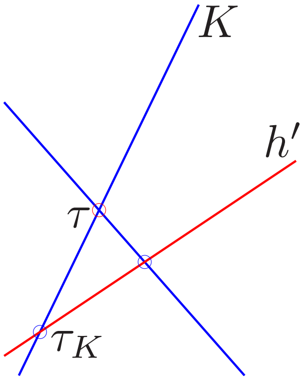
$$g(x) - P_{\mathbb{H}} g(x) = \sum_{K\in\binom{\mathbb{H}}{d-1}} \Big(\prod_{h\in\mathbb{H}\backslash K} \frac{h(x)}{h_\uparrow(\xi_K)} \Big) \, [\tau, x \mid \xi_K]g$$



error for $k = 1$

$\mathbb{H}' := \mathbb{H} \cup \{h'\}$.



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the last equality holding for $x = \tau_K$, $K \in \binom{\mathbb{H}}{d-1}$, hence

$$g(x) - P_{\mathbb{H}'} g(x) = \sum_{K \in \binom{\mathbb{H}}{d-1}} \Big(\prod_{h \notin K} \frac{h(x)}{h_{\uparrow}(\xi_K)}\Big)\, [\tau, \tau_K, x \mid \xi_K, x - \tau_K] g$$

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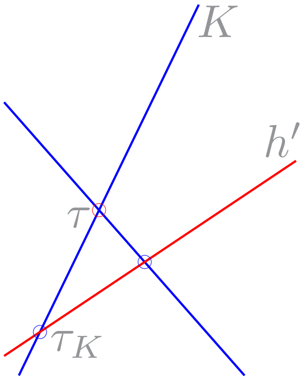

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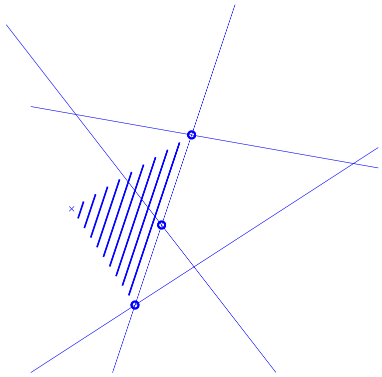
d^2 terms???

$$x - \tau_K = \sum_{K'\in\binom{K\cup\{h'\}}{d-1}} \Big(\prod_{h\in(K\cup\{h'\})\setminus K'} \frac{h(x)}{h_\uparrow(\xi_{K'})}\Big)\, \xi_{K'}$$

Error in Chung-Yao interpolation

$$g(x) - P_{\mathbb{H}} g(x) = \sum_{K\in\binom{\mathbb{H}}{d-1}} p_K(x)\; \underbrace{[\mathrm{T}_K, x \mid \xi_K, \ldots, \xi_K]g}_{k+1\text{st div.dif.}}$$

$d = 2, \quad k = 2$



$$p_K(x) := \prod_{h\in\mathbb{H}\setminus K} \frac{h(x)}{h_\uparrow(\xi_K)},$$

$$\mathrm{T}_K := \mathrm{T}\cap(\cap K)$$

$$0 \neq \xi_K \quad || \quad \cap K := \{x \in \mathbb{R}^d : \; h(x) = 0 \;\; \forall h \in K\}$$

i.e.,

$$h_\uparrow(\xi_K) = 0 \quad \iff \quad h \in K$$

The case $k = 1$ (in different notation and shorter proof) is due to Waldron'98.

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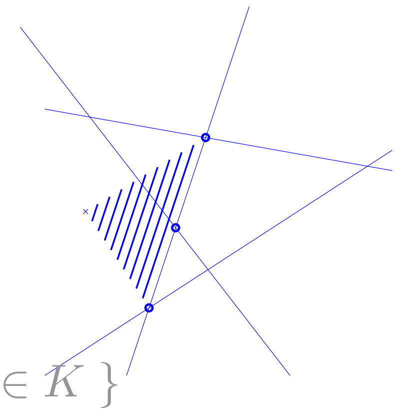

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- $\delta_{\mathrm{T}} p_K = 0$, for all $K \in \binom{\mathbb{H}}{d-1}$ (since $\emptyset \neq K \cap H$, for all $H \in \binom{\mathbb{H}}{d}$).
- **biorthogonality**: $K \neq K' \implies D_{\xi_K}^{k+1} p_{K'} = 0$
  since $D_{\xi_K}^{k+1} \prod_{h \notin K'} h = (k+1)! \prod_{h \notin K'} h_\uparrow(\xi_K)$.

Induction on k

Assuming

$$g(x) - P_{\mathbb{H}} g(x) = \sum_{K \in \binom{\mathbb{H}}{d-1}} p_K(x)\, [\mathrm{T}_K, x \mid \xi_K, \ldots, \xi_K] g$$

to hold for any $\mathbb{H}$ with $\#\mathbb{H} = k + d$, let h' be such that $\mathbb{H}' := \mathbb{H} \cup h' := \mathbb{H} \cup \{h'\}$ is in general position. Then the sites for $\mathbb{H}'$ not in T are

$$\tau_{K \cup h'}, \quad K \in \binom{\mathbb{H}}{d-1},$$

and

$$p_{K'}(\tau_{K \cup h'}) \neq 0 \quad \Longleftrightarrow \quad K' \neq K$$

hence, by induction hypothesis, for $x \in \mathrm{T}' := \{\tau_H : H \in \binom{\mathbb{H}'}{d-1}\}$,

$$g(x) - P_{\mathbb{H}} g(x) = \sum_{K \in \binom{\mathbb{H}}{d-1}} \prod_{h \in \mathbb{H} \setminus K} \frac{h(x)}{h_{\uparrow}(\xi_K)} [\mathrm{T}_K, \tau_{K \cup h'} \mid \xi_K, \ldots, \xi_K] g$$

while the right side is a polynomial of degree $\leq k + 1$, hence

$$g(x) - P_{\mathbb{H}'} g(x) = \sum_{K \in \binom{\mathbb{H}}{d-1}} \prod_{h \in \mathbb{H} \setminus K} \frac{h(x)}{h_{\uparrow}(\xi_K)} [\mathrm{T}_K, \tau_{K \cup h'}, x \mid \xi_K, \ldots, \xi_K, x - \tau_{K \cup h'}] g$$

The proof is finished with the aid of a bivariate divided difference identity.

The Sauer-Xu error formula

Set-up:

- $\mathrm{T} \subset \mathbb{R}^d$, $F = \Pi_{\le k}(\mathbb{R}^d)$, (T, F) correct;
- $(\mathrm{T}_0, \ldots, \mathrm{T}_k)$ a(ny) partition of T such that $(\underbrace{\cup_{0\le i\le j}\mathrm{T}_i}_{=: \mathrm{T}_{\le j}}, \Pi_{\le j})$ is correct for all j;
- P_j the linear projector with range $\Pi_{\le j}$ and nullspace $\ker \delta_{\mathrm{T}_{\le j}}$, all j.
- $\ell_\tau^{[j]}$ the (fundamental) Lagrange polynomial in $\Pi_{\le j}$ for $\tau \in \mathrm{T}_j$, i.e., $\ell_\tau^{[j]}(\tau') = \delta_{\tau,\tau'}$ for all $\tau' \in \mathrm{T}_{\le j}$;

$$g(x) - (P_k g)(x) = \sum_{\tau \in \mathrm{T}_k} \ell_\tau^{[k]}(x) \sum_{\boldsymbol{\sigma} \in \Sigma_{k,\tau}} c_{\boldsymbol{\sigma}}\, [\boldsymbol{\sigma}, x \mid \Delta\boldsymbol{\sigma}, x - \tau] g$$

with
$$\Sigma_{k,\tau} := \mathrm{T}_0 \times \cdots \times \mathrm{T}_{k-1} \times \{\tau\},$$

$$c_{\boldsymbol{\sigma}} := \prod_{j=0}^{k-1} \ell_{\sigma_j}^{[j]}(\sigma_{j+1}),$$

$$\Delta(\sigma_0, \ldots, \sigma_j) := (\sigma_1 - \sigma_0, \ldots, \sigma_j - \sigma_{j-1}).$$

Is it optimal, good, acceptable, true ??? Check it for $k = 0$.

Proof by induction

Assume

$$\begin{aligned} g(x)-(P_j g)(x) &= \sum_{\tau\in\mathrm{T}_j} \ell_\tau^{[j]}(x) \sum_{\boldsymbol{\sigma}\in\Sigma_{j,\tau}} c_{\boldsymbol{\sigma}}\, [\boldsymbol{\sigma}, x \mid \Delta\boldsymbol{\sigma}, x-\tau]g \\ &= \underbrace{\sum_{\tau\in\mathrm{T}_j} \ell_\tau^{[j]}(x) \sum_{\boldsymbol{\sigma}\in\Sigma_{j,\tau}} c_{\boldsymbol{\sigma}}\, [\boldsymbol{\sigma}, y \mid \Delta\boldsymbol{\sigma}, x-\tau]g}_{=:\, G(x,y)} \end{aligned}$$

the last equality holding for any y at $x \in \mathrm{T}_{\le j}$, while $G(\bullet, y) \in \Pi_{\le j+1}$. Hence

$$G(x,y) = \sum_{\tau'\in\mathrm{T}_{j+1}} \ell_{\tau'}^{[j+1]}(x) G(\tau', y). \qquad (*)$$

Applying $\mathrm{id} - P_{j+1}$ to the above equation $g(x) - (P_j g)(x) = G(x,x)$, we get:

$$\begin{aligned} g(x)-(P_{j+1} g)(x) &= G(x,x) - \sum_{\tau'\in\mathrm{T}_{j+1}} \ell_{\tau'}^{[j+1]}(x) G(\tau', \tau') \\ &= \sum_{\tau'\in\mathrm{T}_{j+1}} \ell_{\tau'}^{[j+1]}(x) (G(\tau', x) - G(\tau', \tau')) \end{aligned}$$

(last equality by $(*)$), while

$$\begin{aligned} G_j(\tau', x) - G_j(\tau', \tau') &= [\tau', x \mid x-\tau'] G(\tau', \bullet) \\ &= \sum_{\tau\in\mathrm{T}_j} \ell_\tau^{[j]}(\tau') \sum_{\boldsymbol{\sigma}\in\Sigma_{j,\tau}} c_{\boldsymbol{\sigma}}\, [\tau', x \mid x-\tau'][\boldsymbol{\sigma}, \bullet \mid \Delta\boldsymbol{\sigma}, \tau'-\tau]g \\ &= \sum_{\boldsymbol{\sigma}'\in\Sigma_{j+1,\tau'}} c_{\boldsymbol{\sigma}'}\, [\boldsymbol{\sigma}', x \mid \Delta\boldsymbol{\sigma}', x-\tau']g \end{aligned}$$

What to hope for in an error formula

Set-up:

- $\mathrm{T} \subset \mathbb{F}^d$, $F \subset \Pi(\mathbb{F}^d)$, (T, F) correct.
- $P := P_{\mathrm{T},F}$ the linear projector with $\operatorname{ran} P = F$, $\ker P = \ker \delta_{\mathrm{T}}$.
- Hope for

$$g(x) - (Pg)(x) = \sum_{b \in B} b(x)(I_b g)(x)$$

with B a suitable finite set, and I_b a linear operator depending on b.

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- P is an **ideal** projector, in that $\ker P = \ker \delta_{\mathrm{T}}$ is an ideal, i.e., a linear space of polynomials that is also closed under (pointwise) multiplication. Therefore,

- B must be a basis for the ideal $\ker P$; hope for a ”good” basis; unfortunately, representations in terms of any basis are not unique.

- $\cap_{b \in B} \ker I_b$ must equal F.

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- $\cap_{b \in B} \ker I_b$ must equal F.
- Hope for I_b of the form $C_b q_b(D)$, with C_b linear maps and q_b homogeneous polynomials.

What to hope for in an error formula

- P the linear projector with range F and nullspace $\ker \delta_{\mathrm{T}}$.
- Hope for

$$g(x) - (Pg)(x) = \sum_{b \in B} b(x)(C_b q_b(D) g)(x)$$

with B a (minimal) basis for the ideal $\ker \delta_{\mathrm{T}}$, C_b linear maps, and q_b homogeneous polynomials, so that $F = \cap_{b \in B} \ker q_b(D)$.

What to hope for in an error formula

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with B a (minimal) basis for the ideal $\ker \delta_{\mathrm{T}}$, C_b linear maps, and q_b homogeneous polynomials, so that $F = \cap_{b \in B} \ker q_b(D)$.

- Error formula for Chung-Yao interpolation:

$$g(x) - P_{\mathbb{H}} g(x) = \sum_{K \in \binom{\mathbb{H}}{d-1}} p_K(x) \underbrace{[\mathrm{T}_K, x \mid \xi_K, \ldots, \xi_K]g}_{k+1\text{st div.dif.}}$$

not only lives up to all these hopes, but even has biorthogonality: $q_b(D)b' = 0$ for $b' \in B \backslash \{b\}$.

- Sauer-Xu formula

$$g(x) - (P_k g)(x) = \sum_{\tau \in \mathrm{T}_k} \ell_\tau^{[k]}(x) \sum_{\boldsymbol{\sigma} \in \Sigma_{k,\tau}} c_{\boldsymbol{\sigma}} \, [\boldsymbol{\sigma}, x \mid \Delta \boldsymbol{\sigma}, x - \tau]g$$

???

least interpolation

Set-up:

- $\mathrm{T} \subset \mathbb{F}^d$ arbitrary, finite, choose F as

$$\Pi_{\mathrm{T}} := \lim_{h\to 0} \mathrm{Exp}_{h\mathrm{T}}, \qquad \mathrm{Exp}_{\Sigma} := \mathrm{span}\{\mathrm{e}_\sigma : \sigma \in \Sigma\},$$

with

$$\mathrm{e}_\sigma : x \mapsto \mathrm{e}^{\overline{\sigma}x}.$$

Then (T, F) is correct; in fact,

$$\deg Pg \le \deg g, \qquad q \in \Pi,$$

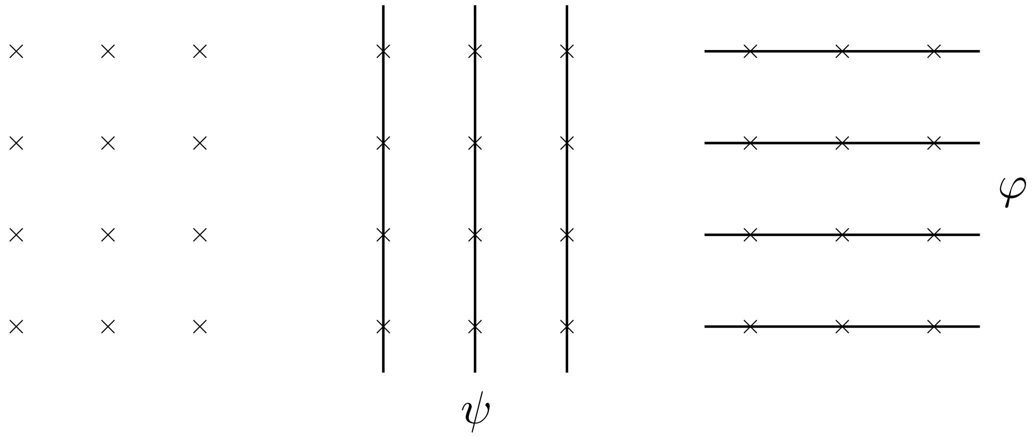
and

$$F = \bigcap_{p\in B} \ker \overline{p}_{\uparrow}(D),$$

with B any basis for the ideal $\ker \delta_{\mathrm{T}}$.

- error formula???

example: bivariate tensor product



$$P_{\mathrm{T}} \;=\; Q \otimes R$$

$$\mathrm{id} \;-\; Q{\otimes}R \;=\; \underbrace{(\mathrm{id} - Q) \otimes R}_{D_1^3} + \underbrace{Q \otimes (\mathrm{id} - R)}_{D_2^4} + \underbrace{(\mathrm{id} - Q) \otimes (\mathrm{id} - R)}_{D_1^3 D_2^4}$$

$$\ker \delta_{\mathrm{T}} \;=\; \mathrm{ideal}(\psi, \varphi)$$

$$\begin{aligned} g(x) - (Q \otimes R)g(x) \;&=\; \psi(x) \int M_1(\bullet|\mathrm{T}, x)\, \psi_{\uparrow}(D)g \\ &\quad+\; \varphi(x) \int M_2(\bullet|\mathrm{T}, x)\, \varphi_{\uparrow}(D)g \end{aligned}$$

A counterexample (B. Mößner, U. Reif)

Set-up:

- Consider bicubic tensor product interpolation as the mesh coalesces at the origin, i.e., $F = \Pi_{\le(3,3)} \subset \Pi(\mathbb{R}^2)$, P with $\operatorname{ran} P = F$ and $\ker P = \cap_{q\in F} \ker \delta_0 q(D)$.
- This is still an ideal projector, as $\ker P = \operatorname{ideal}(x^4, y^4)$.
- But an error formula involving only the pure derivatives $D_1^4 g$ and $D_2^4 g$ cannot hold.

Example

- $f_m(x,y) := m x^3 y^3 (1 - \exp(-m(x^2+y^2)))$, $m = 1, 2, \ldots$ is in $\ker P$, and $\lim_{m\to\infty} |f_m(x,y)| = \infty$ for every $xy \neq 0$;
- $f_m(x,y) = m(xy)^3 - m(xy)^3 \exp(-m(x^2+y^2))$, hence $D_1^4 f_m$ and $D_2^4 f_m$ have the factor $m \exp(-m(x^2+y^2))$, keeping them bounded.

Precisely, $D_1^4 f_m(x,y) = g(x\sqrt{m}, y\sqrt{m})$ with

$$g(u,v) := 4uv^3 \exp(-u^2-v^2)(-30 + 75u^2 - 36u^4 + 4u^6)$$

bounded in absolute value on $\mathbb{R}^2$ by 11, and, by symmetry, $D_2^4 f_m(x,y) = g(y\sqrt{m}, x\sqrt{m})$.

FINIS

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standard pairing: $A_0 \times \Pi : (g,p) \mapsto \langle g,p\rangle := \sum_\alpha \overline{\widehat{g}(\alpha)}\alpha!\widehat{p}(\alpha) \; = \; p(D)\overline{g}(0)$

[BR91-92]. $\Pi_{\mathrm{T}} = \cap_{p|_{\mathrm{T}}=0} \ker \overline{p}_\uparrow(D)$.

Proof:

$$\begin{aligned} p|_{\mathrm{T}} = 0 \quad &\Longrightarrow \quad \mathrm{Exp}(\mathrm{T}) \perp p \\ &\Longrightarrow \quad \Pi_{\mathrm{T}} \perp p_\uparrow \qquad\qquad \langle g,p\rangle = 0 \implies \langle g_\downarrow, p_\uparrow\rangle = 0 \\ &\Longrightarrow \quad \forall\{g \in \Pi_{\mathrm{T}}\}\; \overline{p}_\uparrow(D)g(0) = 0 \\ &\Longrightarrow \quad \forall\{g \in \Pi_{\mathrm{T}}\}\forall\{\alpha\}\; \overline{p}_\uparrow(D)D^\alpha g(0) = 0 \\ &\Longrightarrow \quad \forall\{g \in \Pi_{\mathrm{T}}\}\; \overline{p}_\uparrow(D)g = 0 \end{aligned}$$

standard pairing: $A_0 \times \Pi : (g,p) \mapsto \langle g,p\rangle := \sum_\alpha \overline{\widehat{g}(\alpha)}\alpha!\widehat{p}(\alpha) \ = \ p(D)\overline{g}(0)$

[BR91-92]. $\Pi_{\mathrm{T}} = \cap_{p|_{\mathrm{T}}=0} \ker \overline{p}_{\uparrow}(D) \ =: \ K$.

Proof:

$$\begin{aligned}
p|_{\mathrm{T}} = 0 \quad &\Longrightarrow \quad \mathrm{Exp}(\mathrm{T}) \perp p \\
&\Longrightarrow \quad \Pi_{\mathrm{T}} \perp p_{\uparrow} \qquad\qquad \langle g,p\rangle = 0 \quad \Longrightarrow \quad \langle g_{\downarrow}, p_{\uparrow}\rangle = 0 \\
&\Longrightarrow \quad \forall\{g \in \Pi_{\mathrm{T}}\}\ \overline{p}_{\uparrow}(D)g(0) = 0 \\
&\Longrightarrow \quad \forall\{g \in \Pi_{\mathrm{T}}\}\forall\{\alpha\}\ \overline{p}_{\uparrow}(D)D^{\alpha}g(0) = 0 \\
&\Longrightarrow \quad \forall\{g \in \Pi_{\mathrm{T}}\}\ \overline{p}_{\uparrow}(D)g = 0
\end{aligned}$$

Hence $\Pi_{\mathrm{T}} \subseteq K$.

$$\begin{aligned}
\dim \Pi_{\mathrm{T}} < \infty \quad &\Longrightarrow \quad k := \max\{\deg f : f \in \Pi_{\mathrm{T}}\} < \infty \\
&\Longrightarrow \quad \forall\{|\alpha| = k+1\}\ (()^{\alpha} - P_{\mathrm{T}}()^{\alpha})_{\uparrow} = ()^{\alpha} \\
&\Longrightarrow \quad K \subset \cap_{|\alpha|=k+1} \ker D^{\alpha} = \Pi_{\le k} \subset \Pi
\end{aligned}$$

Hence, $g \in K$ implies that $\Pi \ni \ p := g - P_{\mathrm{T}}g \in K \subset \ker \overline{p}_{\uparrow}(D)$, i.e., $\overline{p}_{\uparrow}(D)p = 0$, hence $p = 0$, so $g \in \Pi_{\mathrm{T}}$.