Twin problems in Approximation and Algebraic geometry.

Quintuplets: Linear algebra, PDEs, Commutative algebra

Approximation theory (AT)

A projector *P* on $\mathbb{C}[\mathbf{x}] \coloneqq \mathbb{C}[x_1,\ldots,x_d]$ is a linear idempotent operator $P:\mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]$

$$\operatorname{ran} P \oplus \ker P = \mathbb{C}[\mathbf{x}]$$

Let $G = \operatorname{ran} P \subset \mathbb{C}[\mathbf{x}]$ be *N*-dimensional and $\mathfrak{g} \coloneqq \{g_1,\ldots,g_N\}$ be a basis for *G*.

$\mathbb{C}'[\mathbf{x}] \coloneqq \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[\mathbf{x}],\mathbb{C})$ is the algebraic dual of $\mathbb{C}[\mathbf{x}]$.

Any projector *P* onto *G*: $P = \sum_{k=1}^{N} \lambda_k \otimes g_k$

$$Pf = \sum \lambda_k(f) g_k,\ \lambda_k(f) = \lambda_k(Pf),\ \forall f \in \mathbb{C}[\mathbf{x}]$$

$$\lambda_k \in \mathbb{C}'[\mathbf{x}],\ \lambda_k(g_j) = \delta_{k,j}$$

$$\Lambda \coloneqq \operatorname{span}\{\lambda_k\} = \operatorname{ran} P^* = (\ker P)^{\perp} \subset \mathbb{C}'[\mathbf{x}]$$

$$P = \sum_{k=1}^{N} \lambda_k \otimes g_k$$

$$\Lambda \coloneqq \text{span}\,\{\lambda_k\} = \text{ran}\,P^* = (\ker P)^\perp \subset \mathbb{C}'[\mathbf{x}]$$

Λ is correct for *G* if the interpolation problem : find

$$g \in G : \lambda(f) = \lambda(g), \forall \lambda \in \Lambda$$

has a unique solution for every $f \in \mathbb{C}[\mathbf{x}]$.

If *Λ* is correct for *G*, *then Λ* defines a projector *P* onto *G* by letting *Pf=g*: solution of interpolation problem for f.

1) Projectors *P* onto *G*

2) Subspaces $J \subset \mathbb{C}[\mathbf{x}]$ that complement G:

$$G \oplus J = \mathbb{C}[\mathbf{x}]$$

3) Subspaces $\Lambda \subset \mathbb{C}'[\mathbf{x}]$ that are correct for G

Five Examples: G=span $\{1,x,y\}$

$$P = \lambda_1 \otimes g_1 + \lambda_2 \otimes g_2 + \lambda_3 \otimes g_3$$

$$Tf = f(0)1 + (D_x f)(0)x + (D_y f)(0)y$$

$$P_* f = f(0)1 + (D_x f)(0)x + ((D_y + \tfrac{1}{2}D_x^2)f)(0)y$$

$$Rf = f(0)1 + (D_x f)(0)x + ((D_y + \tfrac{1}{2}D_y^2)f)(0)y$$

$$Lf = f(0)(1 - x - y) + f(1,0)x + f(0,1)y$$

$$Hf = f(0)(1 - y) + (D_x f)(0)(x - y) + f(1,1)(y)$$

$$\Lambda_T = \{\delta_{\mathbf{0}}, \delta_{\mathbf{0}} \circ D_x, \delta_{\mathbf{0}} \circ D_y\}$$

$$\Lambda_{P_*} = \{\delta_{\mathbf{0}}, \delta_{\mathbf{0}} \circ D_x, \delta_{\mathbf{0}} \circ (\tfrac{1}{2}D_x^2 + 2D_y)\}$$

$$\Lambda_R = \{\delta_{\mathbf{0}}, \delta_{\mathbf{0}} \circ D_x, \delta_{\mathbf{0}} \circ (\tfrac{1}{2}D_y^2 + 2D_y)\}$$

$$\Lambda_L = \{\delta_{\mathbf{0}}, \delta_{(0,1)}, \delta_{(1,0)}\}$$

$$\Lambda_H = \{\delta_{\mathbf{0}}, \delta_{\mathbf{0}} \circ D_x, \delta_{(1,1)}$$

$$J_T = \{f : f(\mathbf{0}) = (D_x f)(0) = (D_y f)(0) = 0\}$$

$$J_{P_*} = \{f : f(\mathbf{0}) = (D_x f)(0) = (D_y + \tfrac{1}{2}D_x^2)f(0) = 0\}$$

$$J_R = \{f : f(\mathbf{0}) = (D_x f)(0) = (D_y + \tfrac{1}{2}D_y^2)f(0) = 0\}$$

$$J_L = \{f : f(\mathbf{0}) = f(1,0) = f(0,1) = 0\}$$

$$J_H = \{f : f(\mathbf{0}) = (D_x f)(0) = f(1,1) = 0\}$$

Definition (G. Birkhoff) A projector is called ideal if

$\ker P$ is and ideal in $\mathbb{C}[\mathbf{x}]$.

$\mathfrak{P}_G$ is the family of all ideal projectors onto *G*:

Lagrange (interpolation) projectors and Taylor projectors are ideal projectors

Taylor projector: $T : \sum c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \to \sum_{|\boldsymbol{\alpha}| \le n} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$

$$\ker T = \sum_{|\boldsymbol{\alpha}| > n} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$$

Lagrange (interpolating) projector at sites: $\mathbf{z}_1, \ldots, \mathbf{z}_N \in \mathbb{C}^d$

$$(Pf)(\mathbf{z}_j) = f(\mathbf{z}_j), \ \forall f \in \mathbb{C}[\mathbf{x}]$$

$$\ker P = \{f \in \mathbb{C}[\mathbf{x}] : f(\mathbf{z}_1) =, \ldots, = f(\mathbf{z}_N) = 0\}$$

In one variable (d=1) ideal projectors are the Hermite interpolation projectors.

$$\Lambda \coloneqq \text{span}\,\{\lambda_k\} = \text{ran}\, P^* = (\ker P)^\perp \subset \mathbb{C}'[\mathbf{x}]$$

What Λ define an ideal projector onto G?

Duality

$$\mathbb{C}_0^\infty \ni (a_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{Z}_+^d} \sim f = \sum a_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{C}[\mathbf{x}]$$

$$\mathbb{C}'[\mathbf{x}] \sim \mathbb{C}^\infty = \{(c_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{Z}_+^d}\}$$

$$(c_{\boldsymbol{\alpha}}) \sim \hat{\lambda} \in \mathbb{C}'[\mathbf{x}]$$

$$\widehat{\lambda}(f) \coloneqq \sum \bar{c}_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}}$$

$$\widehat{\lambda}(f) \coloneqq \sum w_{\boldsymbol{\alpha}} \bar{c}_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \qquad \widehat{\lambda}(f) \coloneqq \sum (\boldsymbol{\alpha}!) \bar{c}_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}}$$

Duality (DE)

$$\widehat{\lambda}(f) \coloneqq \sum (\boldsymbol{\alpha}!)\bar{c}_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}}$$

$$\mathbb{C}'[\mathbf{x}] \sim \mathbb{C}[[\mathbf{x}]] \coloneqq \{\lambda = \sum c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}\}$$

Ring,
algebra,
module

$$\mathbb{C}'[\mathbf{x}] = \mathbb{C}[[\mathbf{x}]],\ \mathbb{C}[\mathbf{x}]$$

$$\mu_i : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}],\ \mu_i(\lambda) = x_i \cdot \lambda$$

$$D_i : \mathbb{C}[[\mathbf{x}]] \to \mathbb{C}[[\mathbf{x}]]$$

$$\boxed{\widehat{D_i\lambda}(f) = \widehat{\lambda}(x_i \cdot f)} \qquad \boxed{(\mu_{x_i})^* = D_i}$$

E. Fischer

F. Macauley

$$\mathbb{C}'[\mathbf{x}] \sim \mathbb{C}[[\mathbf{x}]] \coloneqq \{\lambda = \sum c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}\}$$

$$\hat{\lambda}(f) = \pi^{-d} \int \bar{\lambda}(\mathbf{x}) f(\mathbf{x}) e^{-|\mathbf{x}|^2} dV = \pi^{-d} \sum \int \bar{c}_{\boldsymbol{\alpha}} \bar{\mathbf{x}}^{\boldsymbol{\alpha}} f(\mathbf{x}) e^{-|\mathbf{x}|^2} dV$$

E. Fischer

Macauley

$$\hat{\lambda}(f) \coloneqq (\lambda(D_1, \ldots, D_d) f)(\mathbf{0})$$

$$\widehat{(x^3 + 3y^2 + 1)}(f) = (D_x^3 f + 3D_y^2 f + f)(\mathbf{0})$$

$$\delta_{\mathbf{z}}(f) \coloneqq f(\mathbf{z}) = \widehat{(e^{\mathbf{z} \cdot \mathbf{x}})}(f)$$

$$\mathbb{C}'[\mathbf{x}] \sim \mathbb{C}[[\mathbf{x}]] \coloneqq \{\lambda = \sum c_{\alpha}\mathbf{x}^{\alpha}\}$$

$$\boxed{\widehat{D_i\lambda}(f) = \widehat{\lambda}(x_i \cdot f)} \qquad \boxed{(\mu_{x_i})^* = D_i}$$

Theorem (Macauley) $J \subset \mathbb{C}[\mathbf{x}]$ is an ideal iff

$$J^{\perp} \coloneqq \{\lambda \in \mathbb{C}[[\mathbf{x}]] \; : \widehat{\lambda}(f) = 0, \forall f \in J\}$$

is *D*-invariant, i.e.,

$$\lambda \in J^{\perp} \Rightarrow D_i\lambda \in J^{\perp}, i = 1, \ldots, d$$

(DE): *D*-invariant subspaces are solutions of homogeneous systems of PDEs with constant coefficients

Theorem: $P = \sum_{k=1}^{N} \lambda_k \otimes g_k$ is an ideal projector iff $\Lambda = (\ker P)^{\perp}$ is *D*-invariant.

(AT) Ideal projectors onto *G*

(DE) *D*-invariant subspaces correct for *G* : $\mathfrak{D}_G$

5 Examples: G=span $\{1,x,y\}$

$$P = \lambda_1 \otimes g_1 + \lambda_2 \otimes g_2 + \lambda_3 \otimes g_3$$

$Tf = f(0)1 + (D_x f)(0)x + (D_y f)(0)y \qquad \Lambda_T = \text{sp}\,\{1, x, y\}$

$P_* f = f(0)1 + (D_x f)(0)x + ((D_y + \frac{1}{2}D_x^2)f)(0)y \qquad \Lambda_{P_*} = \text{sp}\,\{1, x, y + \frac{1}{2}x^2\}$

$Rf =$ ~~$f(0)1 + (D_x f)(0)x + ((D_y + \frac{1}{2}D_y^2)$~~$f)(0)y \qquad \Lambda_R = \text{sp}\,\{1, x, y + \frac{1}{2}y^2\}$

$Lf = f(0)(1 - x - y) + f(1,0)x + f(0,1)y \qquad \Lambda_L = \text{sp}\,\{1, e^x, e^y\}$

$Hf = f(0)(1 - y) + (D_x f)(0)(x - y) + f(1,1)(y) \qquad \Lambda_H = \text{sp}\,\{1, x, e^{x+y}\}$

Commutative algebra (CA)

studies ideals $J \subset \mathbb{C}[\mathbf{x}]$ and associated affine varieties:

$$\mathcal{Z}(J) \coloneqq \{\mathbf{z} \in \mathbb{C}^d : f(\mathbf{z}) = 0, \forall f \in J\}$$

$$G \oplus \ker P = \mathbb{C}[\mathbf{x}] \qquad J = \ker P \Leftrightarrow G \oplus J = \mathbb{C}[\mathbf{x}]$$

In the language of CA, *G* spans the finite-dimensional algebra $G \simeq A = \mathbb{C}[\mathbf{x}]/J$

The set of all ideals *J* with $G \simeq A = \mathbb{C}[\mathbf{x}]/J$ is $\mathfrak{J}_G$

$\mathcal{Z}(J)$ is the set of interpolation points (sites) for *P*.

$$\#\mathcal{Z}(J) \leq \dim G = \text{codim}\, J = \text{colength}\, J =: \dim \mathbb{C}[\mathbf{x}]/J$$

$J^{\perp} = \hat{A} = \text{Hom}_{\mathbb{C}}(A, \mathbb{C})$ dualizing module of *A*. The D-invariance gives it a structure of a module over *A*:

$$(a\lambda)(b) = \lambda(ab), \forall a, b \in A$$

(D. Hilbert): Every ideal $J \subset \mathbb{C}[\mathbf{x}]$ is finitely generated:

$$J = \langle h_1, \dots, h_n \rangle : f \in J \Leftrightarrow f = \sum f_j h_j$$

$$J \in \mathfrak{J}_G \Rightarrow J = \ker P = \mathrm{ran}\,(I - P) = \{f - Pf : f \in \mathbb{C}[\mathbf{x}]\}$$

Find $f_j : h_j = f_j - Pf_j$ and you found *P*.

Is there a canonical way of determining nice generators?

$$P : \mathbb{C}[x] \to \mathbb{C}_{<n}[x] \Rightarrow \langle x^n - Px \rangle = \ker P$$

$\mathfrak{g} := \{g_1, \dots, g_N\}$ basis for G

Border of $\mathfrak{g}$:

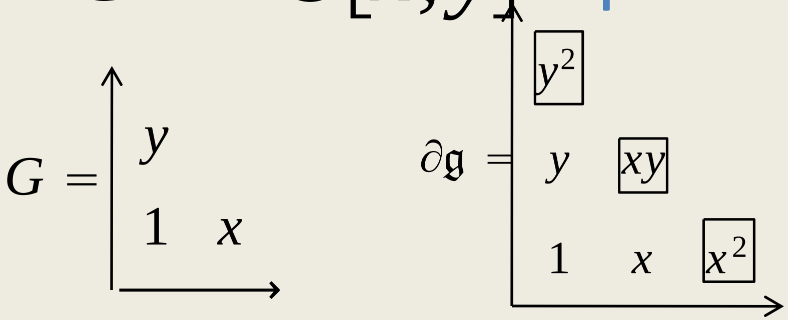
$$\partial\mathfrak{g} := \{1, x_i g_k, i = 1, \dots, d,\ k = 1, \dots, dimG\} \backslash G$$

(AT) $\{Pf, f \in \partial\mathfrak{g}\}$ determines *P*

(CA) $\{f - Pf, f \in \partial\mathfrak{g}\}$ is an (ideal) basis for *J:=ker P*

Border basis:
L. Robbiano,
M. Kreuser,
B. Mourrain,
C. de Boor,
B.S

$G \subset \mathbb{C}[x,y]$ spanned by: $\mathfrak{g} = (1,x,y)$



$$G = \begin{array}{ll} y & \\ 1 & x \end{array} \qquad \partial\mathfrak{g} = \begin{array}{lll} \boxed{y^2} & & \\ y & \boxed{xy} & \\ 1 & x & \boxed{x^2} \end{array}$$

$$\begin{aligned} Px^2 &= a_0 + b_0 x + c_0 y, \\ Pxy &= a_1 + b_1 x + c_1 y, \\ Py^2 &= a_2 + b_2 x + c_2 y. \end{aligned}$$

$$\ker P = \langle x^2 - Px^2, xy - Pxy, y^2 - Py^2 \rangle$$

(AT)

$$Tf = f(0)1 + (D_x f)(0)x + (D_y f)(0)y$$

$$P_* f = f(0)1 + (D_x f)(0)x + ((D_y + \tfrac{1}{2}D_x^2)f)(0)y$$

$$Lf = f(0)(1 - x - y) + f(1,0)x + f(0,1)y$$

$$Hf = f(0)(1 - y) + (D_x f)(0)(x - y) + f(1,1)(y)$$

(DE)

$$\Lambda_T = \mathrm{sp}\{1,x,y\}$$

$$\Lambda_{P_*} = \mathrm{sp}\{1,x,\tfrac{1}{2}x^2 + y\}$$

$$\Lambda_L = \mathrm{sp}\{1 = e^0, e^x, e^y\}$$

$$\Lambda_H = \mathrm{sp}\{1, x, e^{x+y}\}$$

$$T : Tx^2 = T(xy) = Ty^2 = 0,$$

$$P_* : P_* x^2 = y, P_*(xy) = P_* y^2 = 0,$$

$$L : Lx^2 = x, L(xy) = 0, Ly^2 = y,$$

$$H : Hx^2 = Hxy = Hy^2 = y$$

(CA)

$$J_T = \langle x^2, xy, y^2 \rangle$$

$$J_{P_*} = \langle x^2 - y, xy, y^2 \rangle$$

$$J_L = \langle x^2 - x, xy, y^2 - y \rangle$$

$$J_H = \langle x^2, xy, y^2 - y \rangle$$

Linear Algebra (LA)

For an ideal projector *P* onto *G* define

(AT) $\mathbf{M}_P \coloneqq (M_i : G \to G),\ M_i g \coloneqq P(x_i g), g \in G$

de Boor

(CA) $\mathbf{m}_P \coloneqq (m_i : \mathbb{C}[x]/J \to \mathbb{C}[x]/J),\ m_i[f] \coloneqq [x_i f]$

H. Stetter

<u>Theorem</u>: ***M*** is a cyclic sequence of commuting operators: $M_i M_j = M_j M_i$ and

$$g(\mathbf{M})g_0 = g,\ \forall g \in G$$

$g_0 \coloneqq P1$ is a cyclic vector for ***M*:**

$$\{f(\mathbf{M})g_0, f \in \mathbb{C}[x]\} = G$$

$$
\begin{aligned}
Px^2 &= a_0 + b_0 x + c_0 y,\\
Pxy &= a_1 + b_1 x + c_1 y,\\
Py^2 &= a_2 + b_2 x + c_2 y.
\end{aligned}
$$

$$
M_x \coloneqq \left[\begin{array}{cccc}
M_x & 1 & x & y \\
1 & & & \\
x & & & \\
y & & &
\end{array}\right]
$$

$$
M_x \coloneqq \begin{bmatrix} 0 & a_0 & a_1 \\ 1 & b_0 & b_1 \\ 0 & c_0 & c_1 \end{bmatrix}
\qquad
M_y \coloneqq \begin{bmatrix} 0 & a_1 & a_2 \\ 0 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{bmatrix}
$$

$$T : Tx^2 = T(xy) = Ty^2 = 0,$$

$$P_* : P_* x^2 = y, P_*(xy) = P_* y^2 = 0,$$

$$
M_T = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right)
$$

$$
M_{P_*} = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right)
$$

Converse is also true!

(AP),(CA)→ (LA) $P \rightsquigarrow \mathbf{M}_P = (M_1,\dots,M_d)$

(LA)→ (AP),(CA) $\mathbf{L} \coloneqq (L_1,\dots,L_d)$

is a cyclic sequence of *d* commuting *NxN* matrices and *v* is a cyclic vector

$$\begin{array}{rccc}\varphi : & \mathbb{C}[\mathbf{x}] & \to & \mathbb{C}^N \\ & f & \to & f(\mathbf{L})v\end{array}$$

H. Nakajima,
de Boor & B.S

$$\ker\varphi =: J_{\mathbf{L}} = \{f \in \mathbb{C}[\mathbf{x}] : f(\mathbf{L})v = 0\} = \{f \in \mathbb{C}[\mathbf{x}] : f(\mathbf{L}) = 0\}$$

is an ideal of colengh (codimension) *N*. If $G \oplus J_{\mathbf{L}} = \mathbb{C}[\mathbf{x}$
then

$$P_{\mathbf{L}} = (\varphi_{|G})^{-1}\varphi$$

de Boor & B.S

is an ideal projectors onto *G* and

$$\mathbf{L} \sim \mathbf{M}_P,\ \mathcal{Z}(J_{\mathbf{L}}) = \sigma(\mathbf{L})$$

4 Examples: G=span $\{1,x,y\}$

$$Tf = f(0)1 + (D_x f)(0)x + (D_y f)(0)y$$

$$P_* f = f(0)1 + (D_x f)(0)x + ((D_y + \tfrac{1}{2}D_x^2)f)(0)y$$

$$Lf = f(0)(1-x-y) + f(1,0)x + f(0,1)y$$

$$Hf = f(0)(1-y) + (D_x f)(0)(x-y) + f(1,1)(y)$$

$$J_T = \langle x^2, xy, y^2 \rangle$$

$$J_{P_*} = \langle x^2 - y, xy, y^2 \rangle$$

$$J_L = \langle x^2 - x, xy, y^2 - y \rangle$$

$$J_H = \langle x^2, xy, y^2 - y \rangle$$

$$\Lambda_T = \text{sp}\{1, x, y\}$$

$$\Lambda_{P_*} = \text{sp}\{1, x, \tfrac{1}{2}x^2 + y\}$$

$$\Lambda_L = \text{sp}\{1 = e^0, e^x, e^y\}$$

$$\Lambda_H = \text{sp}\{1, x, e^{x+y}\}$$

$$M_T = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right)$$

$$M_{P_*} = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right)$$

$$M_L = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \right)$$

$$M_H = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right)$$

(LA) ➔ (DE) $\lambda \in \Lambda = \mathrm{ran}\, P^*$

$$(M_i^*\lambda)(g)) = \lambda(M_i(g)) = \lambda(P(x_i g))$$

$$= (P^*\lambda)(x_i g) = \lambda(x_i g) = \lambda(\mu_i g)$$

$$\boxed{(\mu_{x_i})^* = D_i} \quad ➔ \quad \boxed{M_i^* = D_{i\,|\,\Lambda}}$$

$$P = \sum_{k=1}^{N} \lambda_k \otimes g_k$$

$$\lambda = \sum b_k \lambda_k \Rightarrow D_i\lambda = M_i^* \begin{pmatrix} b_1 & \dots & b_N \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{pmatrix}$$

The rules of differentiation on Λ define Λ!

Algebraic geometry (AG)

Can I define an ideal projector P by

$$\begin{aligned} Px^2 &= a_0 + b_0 x + c_0 y, \\ Pxy &= a_1 + b_1 x + c_1 y, \\ Py^2 &= a_2 + b_2 x + c_2 y. \end{aligned}$$

with any $(a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2) \in \mathbb{C}^9$?

Does the ideal

$$\langle x^2 - p_0, xy - p_1, y^2 - p_2 \rangle$$

complement G with any linear polynomials p_0, p_1, p_2 ?

In other words, for what $(g_b \in G, b \in \partial\mathfrak{g})$

$$J \coloneqq \langle b - g_b \in G, b \in \partial\mathfrak{g} \rangle \in \mathfrak{I}_G?$$

$$
\begin{aligned}
Px^2 &= a_0 + b_0 x + c_0 y, \\
Pxy &= a_1 + b_1 x + c_1 y, \\
Py^2 &= a_2 + b_2 x + c_2 y.
\end{aligned}
$$

$$
M_x \coloneqq \begin{bmatrix} M_x & 1 & x & y \\ 1 & & & \\ x & & & \\ y & & & \end{bmatrix}
$$

$$
M_x \coloneqq \begin{bmatrix} 0 & a_0 & a_1 \\ 1 & b_0 & b_1 \\ 0 & c_0 & c_1 \end{bmatrix} \qquad M_y \coloneqq \begin{bmatrix} 0 & a_1 & a_2 \\ 0 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{bmatrix}
$$

$$
M_x M_y - M_y M_x = \begin{bmatrix} 0 & a_0 b_1 - a_1 b_0 + a_1 c_1 - a_2 c_0 & a_0 b_2 - a_1 b_1 + a_1 c_2 - a_2 c_1 \\ 0 & a_1 + b_1 c_1 - b_2 c_0 & -b_1^2 + c_2 b_1 + a_2 + b_0 b_2 - b_2 c_1 \\ 0 & c_1^2 - b_0 c_1 - a_0 + b_1 c_0 - c_0 c_2 & b_2 c_0 - b_1 c_1 - a_1 \end{bmatrix}
$$

$$
g(\mathbf{M}) g_0 = g, \ \forall g \in G \ \checkmark
$$

$$
\begin{aligned}
a_0 &= -b_0 c_1 + c_1^2 + b_1 c_0 - c_0 c_2, \\
a_1 &= b_2 c_0 - b_1 c_1, \\
a_2 &= b_1^2 - c_2 b_1 - b_0 b_2 + b_2 c_1.
\end{aligned}
$$

$$M_xM_y - M_yM_x = \begin{bmatrix} 0 & a_0b_1 - a_1b_0 + a_1c_1 - a_2c_0 & a_0b_2 - a_1b_1 + a_1c_2 - a_2c_1 \\ 0 & a_1 + b_1c_1 - b_2c_0 & -b_1^2 + c_2b_1 + a_2 + b_0b_2 - b_2c_1 \\ 0 & c_1^2 - b_0c_1 - a_0 + b_1c_0 - c_0c_2 & b_2c_0 - b_1c_1 - a_1 \end{bmatrix}$$

$$a_0b_1 - a_1b_0 + a_1c_1 - a_2c_0 = 0$$

$$a_1 + b_1c_1 - b_2c_0 = 0$$

$$c_1^2 - b_0c_1 - a_0 + b_1c_0 - c_0c_2 = 0$$

$$a_0b_2 - a_1b_1 + a_1c_2 - a_2c_1 = 0$$

$$-b_1^2 + c_2b_1 + a_2 + b_0b_2 - b_2c_1 = 0$$

$$b_2c_0 - b_1c_1 - a_1 = 0$$

$$\mathcal{B}_g \coloneqq \{(a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2) \in \mathbb{C}^9 \ : \quad \}$$

The border scheme

The border scheme

$$M_i g_k = \begin{cases} x_i g_k & \text{if } x_i g_k \in G \\ p_{x_i g_k} & \text{if } x_i g_k \notin G \end{cases}$$

The coefficients of polynomials $p_{x_i g_k}$ that satisfy a system of polynomial equations:

$$M_i M_j = M_j M_i$$
$$g(\mathbf{M}) g_0 = g, \ \forall g \in G$$

is called a border scheme.

A projector $P \in \mathfrak{P}_G$ (ideal, D-invariant space, cyclic sequence of commuting matrices) is identified with a point $P \in \mathcal{B}_g$

$\mathcal{B}_g$ parametrizes $\mathfrak{P}_G\,(\mathfrak{I}_G, \Lambda_G)$

4 Examples: *G*=span *{1,x,y}*

$$Tf = f(0)1 + (D_x f)(0)x + (D_y f)(0)y$$

$$P_* f = f(0)1 + (D_x f)(0)x + ((D_y + \tfrac{1}{2}D_x^2)f)(0)y$$

$$Lf = f(0)(1 - x - y) + f(1,0)x + f(0,1)y$$

$$Hf = f(0)(1 - y) + (D_x f)(0)(x - y) + f(1,1)(y)$$

$$J_T = \langle x^2, xy, y^2 \rangle$$

$$J_{P_*} = \langle x^2 - y, xy, y^2 \rangle$$

$$J_L = \langle x^2 - x, xy, y^2 - y \rangle$$

$$J_H = \langle x^2, xy, y^2 - y \rangle$$

$$\Lambda_T = \text{sp}\{1, x, y\}$$

$$\Lambda_{P_*} = \text{sp}\{1, x, \tfrac{1}{2}x^2 + y\}$$

$$\Lambda_L = \text{sp}\{1 = e^0, e^x, e^y\}$$

$$\Lambda_H = \text{sp}\{1, x, e^{x+y}\}$$

$$T : Tx^2 = T(xy) = Ty^2 = 0,$$

$$P_* : P_* x^2 = y, P_*(xy) = P_* y^2 = 0,$$

$$L : Lx^2 = x, L(xy) = 0, Ly^2 = y,$$

$$H : Hx^2 = Hxy = Hy^2 = y$$

$$T \sim (0,0,0,0,0,0,0,0,0,) \in \mathfrak{P}_G$$

$$L \sim (0,1,0,0,0,0,0,0,1,) \in \mathfrak{P}_G$$

$$P_* \sim (0,0,1,0,0,0,0,0,0,) \in \mathfrak{P}_G$$

$$H \sim (0,0,1,0,0,1,0,0,1,) \in \mathfrak{P}_G$$

$$\mathbf{L}_T = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right)$$

$$\mathbf{L}_{P_*} = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right)$$

$$\mathbf{L}_L = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \right)$$

$$\mathbf{L}_H = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right)$$

Topology

$\mathcal{B}_g$ parameterizes $\mathfrak{P}_G\left(\mathfrak{I}_G, \Lambda_G, \mathfrak{L}_G\right)$

(AT) $P(t) \to P \Leftrightarrow P(t)f \to Pf,\ \forall f \in \mathbb{C}[\mathbf{x}]$

(DE) $\Lambda(t) \to \Lambda \Leftrightarrow \forall \lambda \in \Lambda\, \exists \lambda(t) \in \Lambda(t) : \lambda(t)f \to \lambda f$

(LA) $\mathbf{L}(t) \to \mathbf{L}$

(CA) $b - P(t)b \to b - P(t)b, \forall b \in \partial\mathfrak{g}$

(AG) $P(t) \to P$ as points in $\mathcal{B}_g \subset \mathbb{C}^k$

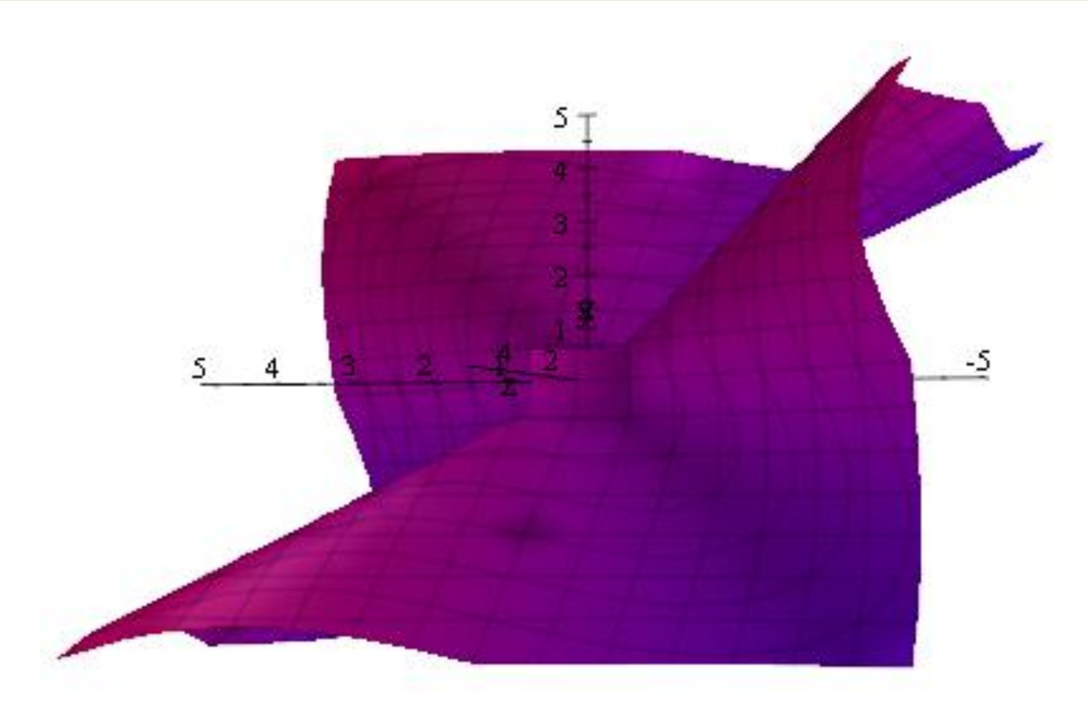
(AG) $P(t) \to P$ as points in $\mathcal{B}_g$ in Zarisski topology

de Boor,
B.S.

de Boor&B.S.

B.S.

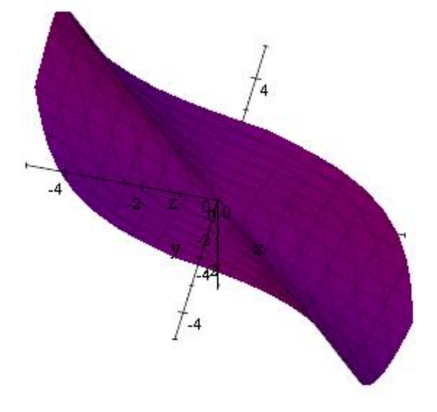
The border scheme



Theorem (J. Fogarty):

For any $G \subset \mathbb{C}[x,y]$ the border scheme is

1) Irreducible
2) Of dimension *2xN=dxN*
3) Smooth
4) Connected



Irreducibility

$J \in \mathfrak{I}_G$ is radical if

$f^m \in J \Rightarrow f \in J$ (CA)

$\#\mathcal{Z}(J) = \dim G,\ \mathcal{Z}(J) = \{\mathbf{z}_1, \ldots, \mathbf{z}_N\}$

P is Lagrange interpolating at $\mathcal{Z}(J)$ (AT)

H. Stetter

$\Lambda = span\{e^{z_j \cdot x}\}$ (DE)

$\mathbf{L} = (L_1, \ldots, L_d)$ are simulteneously diagonalizable (LA)

$Lf = f(0)(1-x-y) + f(1,0)x + f(0,1)y$

$\Lambda_L = \mathrm{sp}\{1 = e^0, e^x, e^y\}$

$J_L = \langle x^2 - x, xy, y^2 - y \rangle$

$L \sim (0,1,0,0,0,0,0,0,1,) \in \mathfrak{P}_G$

$$\mathbf{L}_L = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \right) \sim S \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) S^{-1}$$

Irreducibility

A variety is irreducible if it is not a union of two of its proper subvarieties.

$$\mathcal{L}_{\mathfrak{g}} \coloneqq \{J \in \mathcal{B}_{\mathfrak{g}} : J \text{ is radical}\} \quad \subset \mathcal{B}_{\mathfrak{g}}$$

has a rational parameterization.

$\mathcal{L}_{\mathfrak{g}}^{c}$ is a subvariety, Zarisski closed.

B.S

The closure of $\mathcal{L}_{\mathfrak{g}} : \mathcal{H}_{\mathfrak{g}}$ is irreducible, i.e., not a union of two proper subvarieties, of dimension *dxN*.

$$\mathcal{B}_{\mathfrak{g}} = \mathcal{L}_{\mathfrak{g}}^{c} \cup \mathcal{H}_{\mathfrak{g}}$$

$\mathcal{B}_{\mathfrak{g}}$ is irreducible ⟺ $\mathcal{B}_{\mathfrak{g}} = \mathcal{H}_{\mathfrak{g}}$

$P \in \mathcal{H}_{\mathfrak{g}}$ are called Hermite.

J is radical

P is Lagrange interpolating at $\mathcal{Z}(J)$

$\Lambda = span\{e^{z_j \cdot x}\}$

$\mathbf{L} = (L_1, \ldots, L_d)$ are simulteneously diagonalizable

$P \in \mathcal{L}_{\mathfrak{g}}$

$P(t) \to P \Leftrightarrow P(t)f \to Pf, \forall f \in \mathbb{C}[\mathbf{x}]$

$\Lambda(t) \to \Lambda \Leftrightarrow \forall \lambda \in \Lambda \, \exists \lambda(t) \in \Lambda(t) : \lambda(t)f \to \lambda f$

$b - P(t)b \to b - P(t)b, \forall b \in \partial\mathfrak{g}$

$\mathbf{M}_{P(t)} \to \mathbf{M}_P$

$P(t) \to P$ as points in $\mathcal{B}_g \subset \mathbb{C}^k$

$P(t) \to P$ as points in $\mathcal{B}_g$ in Zarisski topology

$J \in \mathcal{H}_{\mathfrak{g}}$

$P \in \mathcal{P}_G$ is a limit of Lagrange projectors

$\Lambda \in \mathcal{D}_G$ is a limit of pure exponentials

$\mathbf{L} \in \mathfrak{L}_G$: approximable by diagonalizables

Irreducibility

A. Iarrobino,
dim>dxdimG

Theorem: $\mathcal{B}_{\mathfrak{g}}$ is irreducible iff

(AG) $\mathcal{B}_{\mathfrak{g}} = \mathcal{H}_{\mathfrak{g}}$

B.S.

(AP) Every $P \in \mathcal{P}_G$ is a limit of Lagrange projectors

B.S.

(DE) Every $\Lambda \in \mathcal{D}_G$ is a limit of pure exponentials

de
Boor

(LA) Every $\mathbf{L} \in \mathfrak{L}_G$: approximable by diagonalizables

De Boor&B.S.

and irreducible → $\dim \mathcal{B}_{\mathfrak{g}} = d \times \dim G$

Fogarty → True for d=2 …
false for some *G* if *d>2*.

(AP) answers a question of de Boor

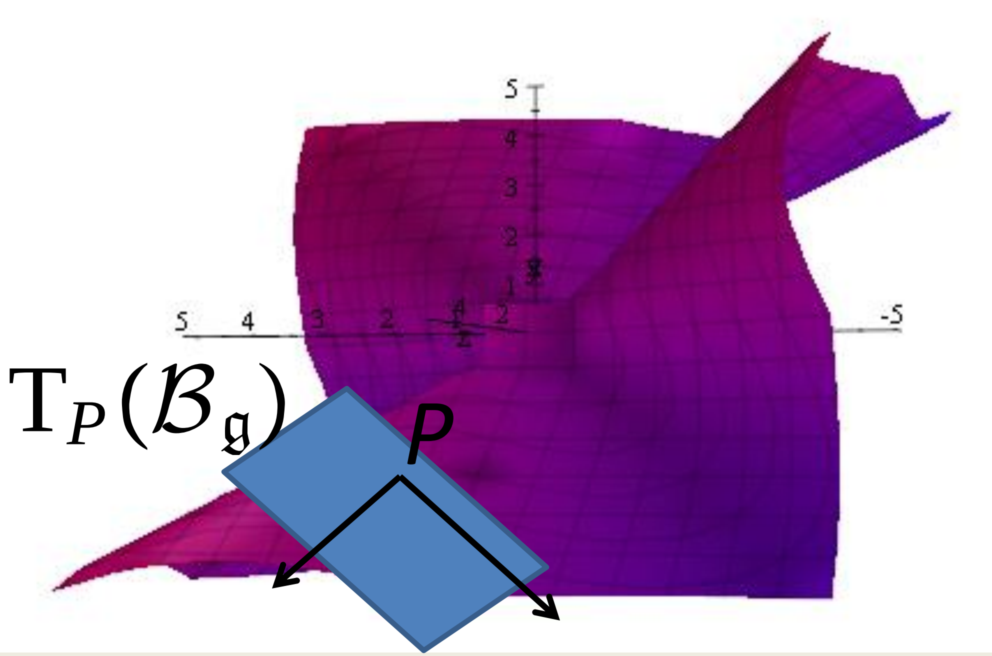
(DE) aswers a question of Lefranc

(LA) answers a question of Guralnick

d=2
Motzkin&
Tausski,
M.Gerstenhaber
R. Guralnick
De Boor&B.S.

Smoothness

$$\dim(\mathrm{T}_P(\mathcal{B}_{\mathfrak{g}})) = \dim(\mathcal{B}_{\mathfrak{g}})_P$$



$$\boxed{\begin{array}{l}\dim(\mathrm{T}_J(\mathcal{B}_{\mathfrak{g}})) = \\ \dim \mathrm{Hom}_{\mathbb{C}[\mathbf{x}]}(J, \mathbb{C}[\mathbf{x}]/J)\end{array}}$$

Miller&Sturmfels

$$S \in \mathrm{Hom}_{\mathbb{C}[\mathbf{x}]}(J, \mathbb{C}[\mathbf{x}]/J) \Rightarrow$$

$$S(fh_1 + gh_2) = [f][Sh_1] + [g][Sh_2]$$

$$T : \mathbb{C}[x,y,z] \to \text{span}\,\{1,x,y,z\}$$

$$J = \{x^2, xy, y^2, yz, z^2, xz\}$$

$$\mathbb{C}[x,y,z]/J = \text{span}\,\{[1],[x],[y],[z]\}$$

[u]=0 for all monomials *u* of degree *>1*

$$Sx^2 = a_0[1] + b_0[x] + c_0[y] + d_0[z]$$

$$Sxy = a_1[1] + b_1[x] + c_1[y] + d_1[z]$$

$$S(fh_1 + gh_2) = [f][Sh_1] + [g][Sh_2]$$

$$S(yx^2) = [y](Sx^2) = [y](a_0[1] + b_0[x] + c_0[y] + d_0[z]) \;\; = a_0[y]$$

$$S(yx^2) = [x](Sxy) = [x](a_1[1] + b_1[x] + c_1[y] + d_1[z]) \;\; = a_1[x]$$

$a_0 = 0, a_1 = 0, \ldots$ *b*-s, *c*-s, and *d*-s are arbitrary

$$\dim(\mathrm{T}_J(\mathcal{B}_{\mathfrak{g}})) = 6 \times 3 = 18 > 12 = d \times \dim G$$

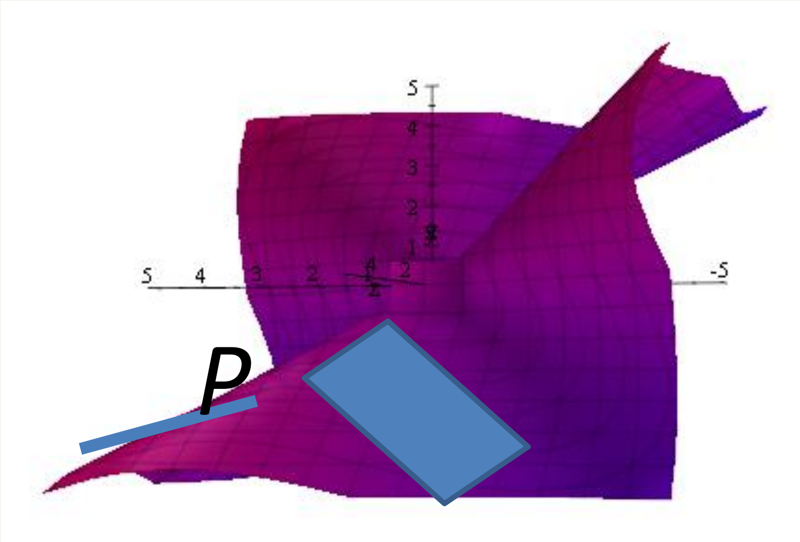
$\mathcal{B}_{\mathfrak{g}}$ is not smooth for *d>2*!!

Emsalem&Iarrobino

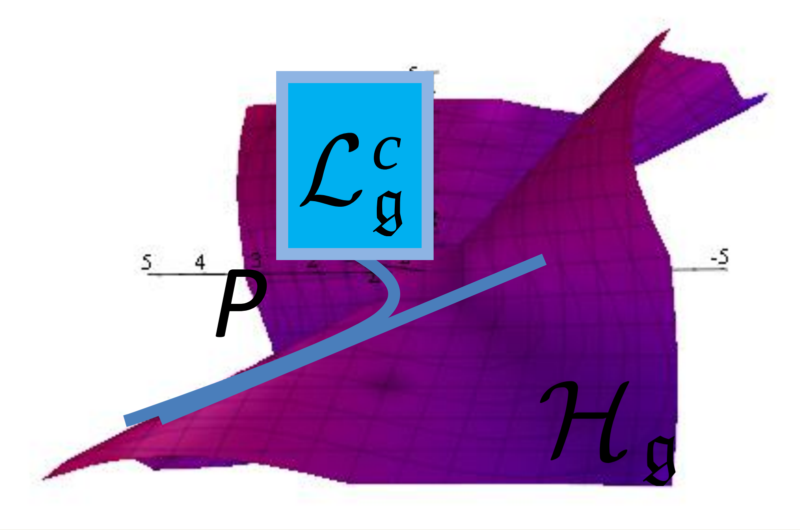
$$P : \mathbb{C}[x,y,z,w] \rightarrow \text{span}\,\{1,x,y,z,w,xz,yz,yw\}$$

$$Pxw = yz, Pu = 0, u \notin G \qquad J = \langle xw - yz, u \notin G\rangle$$

$$\begin{array}{c|cccccccc}
 & 1 & x & y & z & w & xz & yz & yw \\
x^2 & 0 & 2a_2 & 0 & 0 & 0 & \bigstar & \bigstar & \bigstar \\
y^2 & 0 & 0 & 2a_1 & 0 & 0 & \bigstar & \bigstar & \bigstar \\
z^2 & 0 & 0 & 0 & 2a_3 & 0 & \bigstar & \bigstar & \bigstar \\
w^2 & 0 & 0 & 0 & 0 & 2a_4 & \bigstar & \bigstar & \bigstar \\
xy & 0 & a_1 & a_2 & 0 & 0 & \bigstar & \bigstar & \bigstar \\
zw & 0 & 0 & 0 & a_4 & a_3 & \bigstar & \bigstar & \bigstar \\
xw - yz & 0 & a_4 & a_3 & a_1 & a_2 & \bigstar & \bigstar & \bigstar
\end{array}$$



$$\dim(\mathrm{T}_J(\mathcal{B}_{\mathfrak{g}})) \;= \dim\mathrm{Hom}_{\mathbb{C}[\mathbf{x}]}(J, \mathbb{C}[\mathbf{x}]/J) = 25 < d \times \dim G = 32$$



$$Pxw = yz, Pu = 0, u \notin G$$

$$\mathcal{B}_{\mathfrak{g}} = \mathcal{L}_{\mathfrak{g}}^c \cup \mathcal{H}_{\mathfrak{g}}$$

$$P \in \mathcal{L}_{\mathfrak{g}}^c, P \notin \mathcal{H}_{\mathfrak{g}}$$

P is not Hermite!!

Cartwright, Dustin A., Daniel Erman, Mauricio Velasco, Bianca Viray

$\mathcal{B}_{\mathfrak{g}}$ consists of precisely two irreducible components

$$\Lambda = \mathcal{D}\{\lambda_1, \lambda_2, \lambda_3\} \quad (1,x,y,z,w)A_j(1,x,y,z,w)^t = \lambda_j$$

$$\det\begin{bmatrix} 0 & A_1 & -A_2 \\ -A_1 & 0 & A_3 \\ A_2 & -A_3 & 0 \end{bmatrix} \neq 0$$

d=3 $\mathbb{C}[x,y,z]$

No known explicit examples of non-Hermite projectors.

No explicit examples of three commuting matrices that are not approximable by diagonalizables.

Existance of non-Hermite projectors known for *dimG > 101*.

Every ideal projector onto *G* with *dimG <9* is Hermite.

What is the least dimension of G that admits a non-Hermite projector? Sturmefels

There exist three *30x30* matrices that are not approximable by diagonalizables. Guralnick

No such *8x8* matrices. K. Sivic

What is the least size of such matrices? Guralnick

$$P : \mathbb{C}[x,y,z] \to \mathbb{C}[x,y,z]_{\leq 2}$$

False

$Px^3 = yz, Py^3 = xz, Pz^3 = xy, Pu = 0$ is not Hermite.

Problem in (AT) Finally

$$x = \lim\left(\frac{1}{u_1 - u_2} e^{u_1 x} - \frac{1}{u_1 - u_2} e^{u_2 x}\right), u_1, u_2 \to 0$$

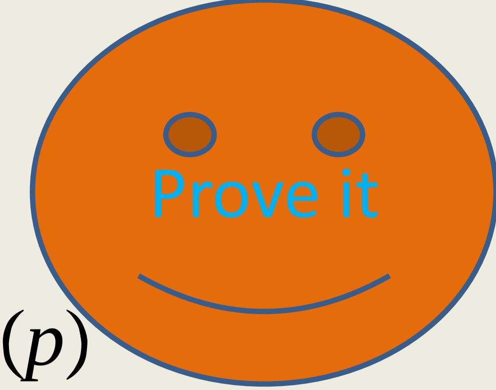
$$\frac{1}{2}x^2 + y = \lim\left(a_1(t)e^{u_1(t)x + v_1(t)y} + a_2(t)e^{u_2(t)x + v_2(t)y} + a_3(t)e^{u_3(t)x + v_3(t)y}\right)$$

$$x^4 + y^4 + z^4 + xyz = \lim\left(\sum_{j=1}^{11} a_j(t)e^{u_j(t)x + v_j(t)y + w_j(t)z}\right)$$

$$= \lim\left(a_{11}(t) + \sum_{j=1}^{10} a_j(t)e^{u_j(t)x + v_j(t)y + w_j(t)z}\right)$$

Theorem: Every polynomial

$$p(x,y,z) = \lim\left(\sum_{j=1}^{N} a_j(t)e^{u_j(t)x+v_j(t)y+w_j(t)z}\right)$$



for some $u_j(t), v_j(t), w_j(t) \to 0,\ N = \dim \mathcal{D}(p)$

An ideal *J* is Gorenstein if $\Lambda = \mathcal{D}(p)$ iff $\mathbf{L}^*$ is cyclic iff

$$\mathbf{L}^* \sim \mathbf{L}$$

$$P_* : P_* x^2 = y, P_*(xy) = P_* y^2 = 0$$

$$\Lambda_{P_*} = \text{span}\{1, x, \tfrac{1}{2}x^2 + y\}$$

$$p(x,y) = \tfrac{1}{2}x^2 + y$$

$$\mathbf{L}_{P_*} = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right)$$

Theorem: in three variables $\mathcal{B}_g^{Gor} \subset \mathcal{B}_g$ is irreducible.

D. Buchsbaum & D. Eisenbud,
S. Diesel…

$$\frac{1}{4}(x^4 + y^4 + z^4) + xyz = \lim\left(\sum_{j=1}^{11} a_j(t) e^{u_j(t)x + v_j(t)y + w_j(t)z}\right.$$

$$= \lim\left(a_{11}(t) + \sum_{j=1}^{10} a_j(t) e^{u_j(t)x + v_j(t)y + w_j(t)z}\right)$$

$$P : \mathbb{C}[x,y,z] \to \mathbb{C}[x,y,z]_{\leq 2}$$

Hermite

$$Px^3 = yz, Py^3 = xz, Pz^3 = xy, Pu = 0$$

$$\frac{1}{4}(x^4 + y^4 + z^4) + xyz = \lim(a_{11}(t) + \sum_{j=1}^{10} a_j(t)e^{u_j(t)x+v_j(t)y+w_j(t)z})$$

$$x^3 + yz = \lim(\sum_{j=1}^{10} (a_j(t)u_j(t))e^{u_j(t)x+v_j(t)y+w_j(t)z})$$

$$y^3 + xz = \lim(\sum_{j=1}^{10} (a_j(t)v_j(t))e^{u_j(t)x+v_j(t)y+w_j(t)z})$$

$$z^3 + xy = \lim(\sum_{j=1}^{10} (a_j(t)w_j(t))e^{u_j(t)x+v_j(t)y+w_j(t)z})$$

$\Lambda = \lim \Lambda(t)$ ← Pure exponentials

Dustin Cartwright

$$Px^3 = a_1xy + a_2xz + a_3yz + a_4x^2 + a_5y^2 + a_6z^2$$

$$Py^3 = b_1xy + b_2xz + b_3yz + b_4x^2 + b_5y^2 + b_6z^2$$

$$Pz^3 = c_1xy + c_2xz + c_3yz + c_4x^2 + c_5y^2 + c_6z^2$$

Hermite

$$Pu = 0$$

Is $\mathcal{B}_g$ connected?

L. Robbiano,
M. Kreuser,

$$P_0, P_1 \in \mathfrak{P}_G, \exists P(t) \in \mathfrak{P}_G : P(0) = P_0, P(1) = P_1$$

$$\mathbf{L}_0, \mathbf{L}_1 \in \mathfrak{L}_G, \exists \mathbf{L}(t) \in \mathfrak{L}_G : \mathbf{L}(0) = \mathbf{L}_0, \mathbf{L}(1) = \mathbf{L}_1$$

$$\Lambda_0, \Lambda_1 \in \Lambda_G, \exists \Lambda(t) \in \Lambda_G : \Lambda(0) = \Lambda_0, \Lambda(1) = P_1$$

If $\deg G \leq \deg \partial \mathfrak{g}$ then yes.

$$\mathbf{x}^\alpha \in \partial \mathfrak{g}, P\mathbf{x}^\alpha = \sum c(\boldsymbol{\beta})\mathbf{x}^\beta, c(\boldsymbol{\beta}) \in \mathcal{B}_g$$

$$P(\tfrac{1}{\varepsilon}\mathbf{x})^\alpha = \sum c(\boldsymbol{\beta})(\tfrac{1}{\varepsilon}\mathbf{x})^\beta, c(\boldsymbol{\beta})(\tfrac{1}{\varepsilon})^\beta \in \mathcal{B}_g$$

$$P_\varepsilon \mathbf{x}^\alpha = \sum c(\boldsymbol{\beta})\varepsilon^{\alpha-\beta}\mathbf{x}^\beta$$

$$P_0 \mathbf{x}^\alpha = \sum\nolimits_{|\alpha|=|\beta|} c(\boldsymbol{\beta})\mathbf{x}^\beta \quad P_\delta \mathbf{x}^\alpha = \sum\nolimits_{|\alpha|=|\beta|} c(\boldsymbol{\beta})\delta\mathbf{x}^\beta \qquad T\mathbf{x}^\alpha = 0$$

Theorem: Hilbert scheme $\mathrm{Hilb}_N(\mathbb{C}^d)$ parametrizing the family of all ideals of colength (codimension) N is connected.

Miller&Sturmfels

$$P_0, P_1 \in \mathfrak{P}_G, \exists P(t) : P(0) = P_0, P(1) = P_1, \ \dim \operatorname{ran} P(t) = N$$

$$\mathbf{L}_0, \mathbf{L}_1 \in \mathfrak{L}_G, \exists \mathbf{L}(t) : \mathbf{L}(0) = \mathbf{L}_0, \mathbf{L}(1) = \mathbf{L}_1, \mathbf{L}(t) = N \times N$$

$$\Lambda_0, \Lambda_1 \in \Lambda_G, \exists \Lambda(t) : \Lambda(0) = \Lambda_0, \Lambda(1) = P_1, \ \dim \Lambda(t) = N$$

$$P(t) \notin \mathfrak{P}_G$$

Thank You

very much

indeed

Thank you

Thank You