Scattered data fitting on the sphere using Bernstein-Bezier spherical splines

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Abstract

I will introduce scattered data fitting problems on the sphere and discuss their applications. I will define Bernstein-Bezier polynomials and describe how spherical splines can solve the problems.

Problem

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. Suppose we are given a set of scattered locations on $\mathbb{S}^2$ along with real numbers associated with these locations. The problem is to find a smooth function interpolating or approximating these data.

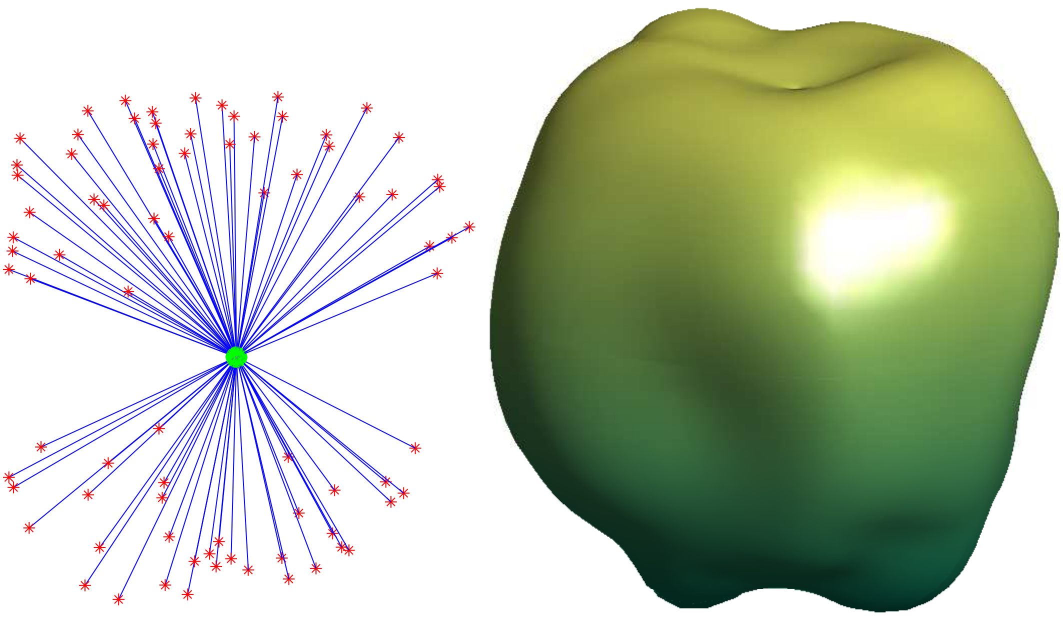

Figure 1: Cubic minimal energy spline interpolating apple data.

Geodesy

$$\begin{cases} \Delta V = 0 & \text{inside, } \mathbb{S}^2 \\ V = f & \text{on the surface of } \mathbb{S}^2. \end{cases}$$

$$V(u) = \frac{1-|u|^2}{4\pi} \int_{\mathbb{S}^2} \frac{f(v)}{|u-v|^3} d\sigma.$$

$$V(u) \approx \frac{1-|u|^2}{4\pi} \int_{\mathbb{S}^2} \frac{s(v)}{|u-v|^3} d\sigma.$$

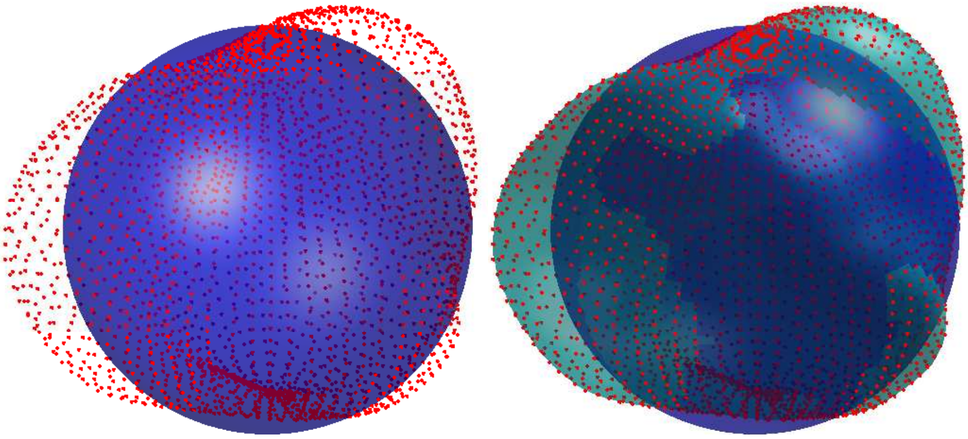

Figure 2: Minimal energy cubic spline interpolating geo-potential boundary data.

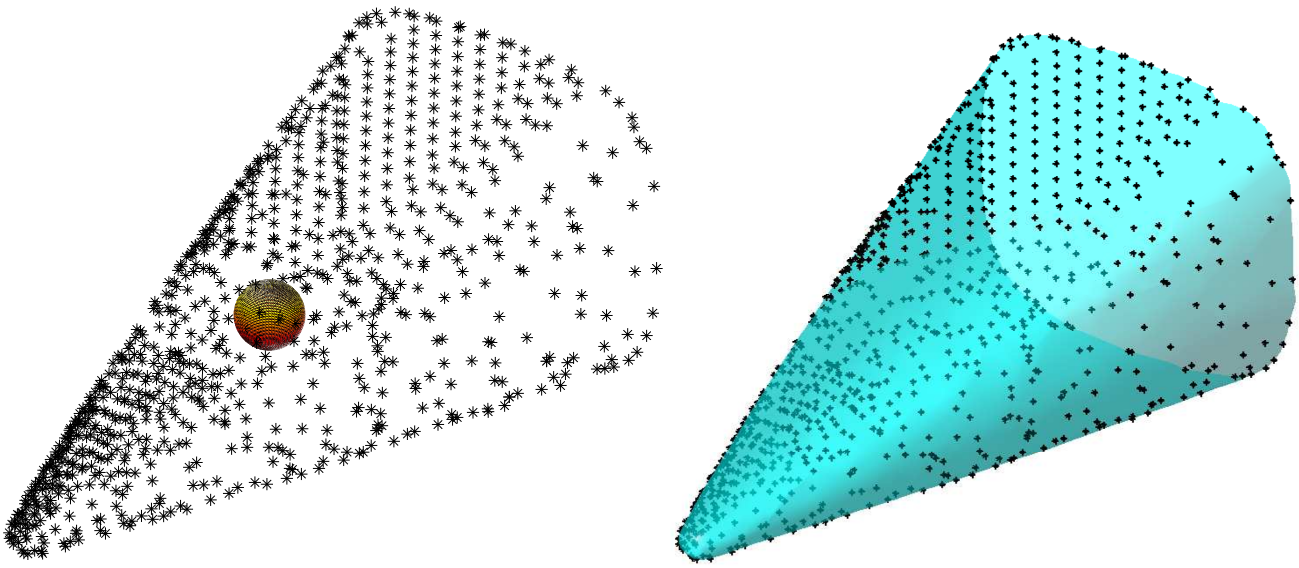

Figure 3: Minimal energy cubic interpolant of the point cloud.

Food Science

$$\begin{cases} u_t(v,t) - \Delta u(v,t) = 0 & v \in B_1(0) \in \mathbb{R}^3, t \in (0,T] \\ u(v,0) = g(v) & v \in B_1(0) \in \mathbb{R}^3 \\ u(v,t) = f(v) \text{ or } \frac{\partial}{\partial r} u(v,t) = f(v) & v \in \mathbb{S}^2 \end{cases}$$

$$\begin{cases} w^2 U(v) + H(v) = \Delta U(v), v \in B_1(0) \\ U(v)|_{\mathbb{S}^2} = F(v) \text{ or } \frac{\partial}{\partial r} U(v)|_{\mathbb{S}^2} = F(v) \end{cases}$$

Spherical PDE

$$\begin{cases} u_t(v,t) - \Delta^* u(v,t) = 0 & v \in \mathbb{S}^2, t \in (0,T] \\ u(v,0) = f(v) & v \in \mathbb{S}^2 \end{cases}$$

$$-\Delta^* U + w^2 U = F$$

Spherical Bernstein-Bezier splines

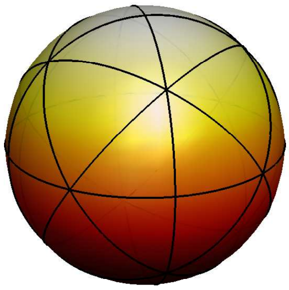

Figure 4: Triangulation on the sphere.

$$\sum_{i=1}^{3} b_i(v)v_i = v$$

$$B^d_{ijk}(v) = \frac{d!}{i!j!k!} b_1(v)^i b_2(v)^j b_3(v)^k, i + j + k = d,$$

$$p = \sum_{i+j+k=d} c_{ijk} B^d_{ijk},$$

$$S^r_d(\Delta) := \{s : s|_\tau \in \mathcal{H}^d(\mathbb{S}^2)\} \cap C^r(\mathbb{S}^2)$$

Global methods

Minimal energy interpolation

$$\Gamma_f := \{s \in S_d^r(\Delta) : s(v) = f(v), \forall v \in \mathcal{V}\}$$

$$\mathcal{E}(f) = \int_{\mathbb{S}^2} \sum_{|\alpha|=2} (D^\alpha f_\delta)^2.$$

$$\mathcal{E}(S_f) = \min_{s \in \Gamma_f} \mathcal{E}(s).$$

Discrete least squares splines

$$\mathcal{L}(s) = \sum_{v \in \mathcal{V}} (s(v) - f(v))^2$$

$$\mathcal{L}(S_f) = \min_{s \in S_d^r(\Delta)} \mathcal{L}(s).$$

Penalized least square fitting

$$\mathcal{P}_\lambda(s) := \mathcal{L}(s) + \lambda\mathcal{E}_\delta(s),$$

$$\mathcal{P}_\lambda(S_f) = \min_{s\in S_d^r(\Delta)}\mathcal{P}_\lambda(s).$$

Linear systems

$$\begin{bmatrix} \mathbf{A} & \mathbf{L}^T \\ \mathbf{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix}$$

$$(\mathbf{A} + \frac{1}{\epsilon}\mathbf{L}^T\mathbf{L})\mathbf{c}^{(\ell+1)} = \mathbf{A}\mathbf{g}\mathbf{c}^{(\ell)} + \frac{1}{\epsilon}\mathbf{L}^T\mathbf{h}$$

Interpolation

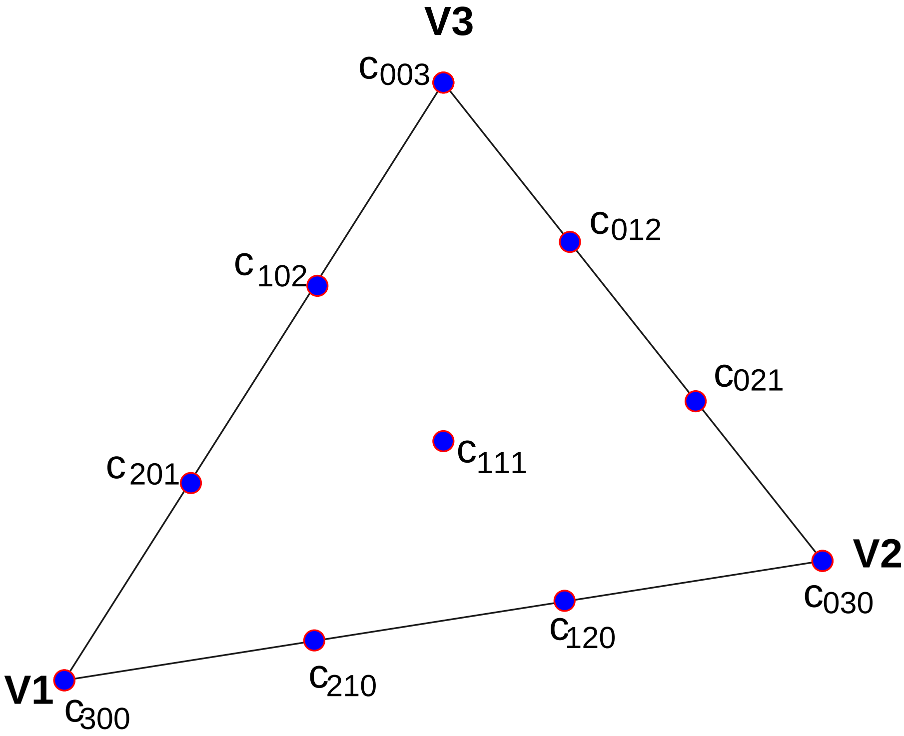

Figure 5: Coefficients of a cubic BB-polynomial.

$$c_{300} = f(v_1)$$

$$c_{030} = f(v_2)$$

$$c_{003} = f(v_3)$$

$$\mathbf{I}\ \mathbf{c} = \mathbf{F}.$$

Smoothness conditions

$$d_{ijk} = \sum_{r+s+t=i} c_{r,j+s,k+t} B^i_{rst}(v_4)$$

$i = 0, ..., r$ and $0 \leq j, k \leq d$ with $i + j + k = d$ (cf.[Alfeld, Neamtu, Schumaker, '96]). Thus a spline $s \in S_d^{-1}(\Delta)$ belongs to $S_d^r(\Delta)$ if and only if

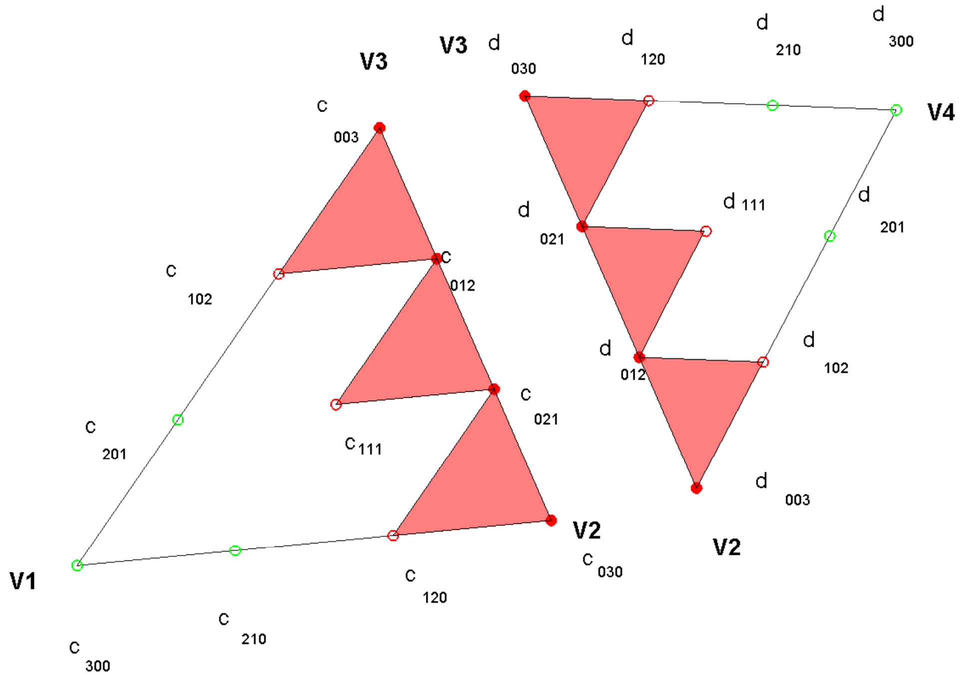

Figure 6: C^1 smoothness conditions.

$$\mathbf{S}\ \mathbf{c} = \mathbf{0}.$$

Energy matrix

$$\mathcal{E}(s) = \int_{\mathbb{S}^2} \sum_{|\alpha|=2} (D^\alpha s_0)^2$$

Minimize $\mathcal{E}(c)$, subject to $\mathbf{Sc} = \mathbf{0}$ and $\mathbf{Ic} = \mathbf{F}$.

$$\mathbf{E}[i,j] := \int_{\mathbb{S}^2} \sum_{|\alpha|=2} (D^\alpha B_i)(D^\alpha B_j).$$

$$\mathbf{E}[i,j] := \sum_{|\alpha|=2} \int_T (D^\alpha B_i)(D^\alpha B_j).$$

Differentiation

$$p_0(v) = p(\frac{v}{|v|}) = |v|^{-d} p(v)$$

$$\begin{aligned} D_{i,j} p_0|_{\mathbb{S}^2} &= (-d)(\delta_{ij} - (d+2)\langle v, e_i\rangle\langle v, e_j\rangle)p(v) \\ &+ (-d)(p_i(v)\langle v, e_j\rangle + p_j(v)\langle v, e_i\rangle) + p_{ij}(v) \end{aligned}$$

$$p_i(v) = b(e_i)^t \nabla_b p(v) \qquad \nabla_b p(v)_i = \frac{\partial}{\partial b_i} p(v)$$

$$p_{ij}(v) = b(e_i)^t H_b p(v) b(e_j) \qquad H_b p(v)_{ij} = \frac{\partial^2}{\partial b_i \partial b_j} p(v)$$

Integration

$$ds = \frac{|v'||N|}{|v'N|}ds', v = \frac{v'}{|v'|}$$

$$\int_T f(v)ds = \int_{T'} f(\frac{v'}{|v'|})\frac{|v'||N|}{|v'N|}ds'$$

$$\int_{T'} g(v')ds' = 2A_{T'}\int_0^1\int_0^{1-b'_2} g(b'_1, b'_2)db'_1 db'_2$$

$$\int_T f(v)ds = 2A_{T'}|N|\int_0^1\int_0^{1-b'_2} f(b'_1, b'_2)\frac{|v'|}{|v'N|}db'_1 db'_2$$

$$\begin{bmatrix} \mathbf{E} & \mathbf{I}^T & \mathbf{S} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{S} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \lambda \\ \eta \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

Theorem 1. Suppose $S_d^r(\Delta)$ is a spline space defined on a β-quasi-uniform triangulation Δ with $|\Delta| \leq 1$ and $d \geq 3r+2$. There exists a constant C depending only on d and β, such that the energy interpolant S_f minimizing $\mathcal{E}$ satisfies

$$\|f - S_f\|_{\infty,\mathbb{S}^2} \leq C|\Delta^2||f|_{2,\infty,\mathbb{S}^2}$$

for all $f \in C^2(\mathbb{S}^2)$.

Theorem 2. Let Δ be a β-quasi-uniform triangulation of the sphere $\mathbb{S}^2$ whose vertices form a subset of the data sites $\mathcal{V}$ and $|\Delta| \leq 1$. Let N be the number of triangles in Δ. Suppose that the data locations $\mathcal{V}$ have the property that for every $s \in S^r_d(\Delta)$ and every $\tau \in \Delta$, there exist a positive constant F_1, independent of s and τ, such that

$$F_1 \|s\|_{\infty,\tau} \leq (\sum_{\mathcal{V}\cap\tau} s(v)^2)^{1/2}. \tag{1}$$

Let F_2 be the largest number of data sites in a triangle $\tau \in \Delta$. That is, we have

$$(\sum_{\mathcal{V}\cap\tau} s(v)^2)^{1/2} \leq F_2 \|s\|_{\infty,\tau}. \tag{2}$$

For $d \geq 3r + 2$ let $s_{\lambda,f}$ be the spline minimizing $\mathcal{P}_\lambda$. Then

$$\|f - s_{\lambda,f}\|_{\infty,\mathbb{S}^2} \leq C|\Delta|^{m+1}|f|_{m+1,\infty,\mathbb{S}^2} + \lambda C'\sqrt{N}$$

for every function f in $W^{m+1,\infty}(\mathbb{S}^2)$. Here m is between 0 and d with $(d - m) \bmod 2 = 0$. The constant C depends on d, β, F_1, F_2, and C' depends on d, β, F_1, F_2, f and $|\Delta|$.

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