OPTIMAL RECOVERY OF CERTAIN CLASSES OF TWICE DIFFERENTIABLE MULTIVARIATE FUNCTIONS

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Let $G \subset \mathbb{R}^d$, $d \in \mathbf{N}$, be a convex body. Denote by $W^2(G)$ the class of functions $f \in C^1(G)$ such that for every unit vector $\mathbf{r} \in \mathbb{R}^d$,

$$\left\| \frac{\partial^2 f}{\partial \mathbf{r}^2} \right\|_G := \operatorname*{ess\,sup}_{\mathbf{x} \in G} \left| \frac{\partial^2 f}{\partial \mathbf{r}^2}(\mathbf{x}) \right| \leq 1.$$

Let $X_n = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subset G$, denote

$$N(f) = N(f, X_n) =$$

$$= (f(\mathbf{x}_1), \ldots, f(\mathbf{x}_n), \text{grad} f(\mathbf{x}_1), \ldots, \text{grad} f(\mathbf{x}_n)) .$$

Let $\Phi_n : \mathbb{R}^{n(d+1)} \to C(G)$ be an arbitrary mapping. Denote

$$S(f) = S(f; X_n, \Phi_n) = \Phi_n (N(f, X_n)) . \tag{1}$$

Error of method (1) over the class $W^2(G)$:

$$R(W^2(G); S) = R(W^2(G); X_n, \Phi_n) = \sup_{f \in W^2(G)} \|f - S(f)\|_G.$$

Problem 1. It is required to find the value

$$R_n(W^2(G)) = \inf_{X_n, \Phi_n} R(W^2(G); X_n, \Phi_n)$$

and optimal algorithms $S(\cdot; X_n^*, \Phi_n^*)$, if they exist.

Known one-dimensional results. Let W_p^r, $r \in \mathbf{N}$, $p \in [1, \infty]$, be the class of functions $f : [0, 1] \to \mathbb{R}$ such that $f^{(r-1)} \in AC$ and $\|f^{(r)}\|_p \leq 1$.

Solution to the one-dimensional problem about optimal recovery of functions from its values at $2n$ points is known on classes W^r_∞ in L_p and W^r_p in L_1, W^rH^ω, ω - concave, in L_1, W^1H^ω in L_q. It follows from results on n-widths by Tikhomirov (1969), Ruban (1975), Makovoz (1979), Pinkus (1979), Ligun (1980).

Ligun, Bojanov, Sun Yong-Sheng. Different settings: Stechkin, Bakhvalov, Subbotin, Arestov, Marchuk and Osipenko, Grebennikov and Morozov.

More references can be found in the books by

Tikhomirov (1976); Traub and Wozniakowski (1980); Korneichuk (1984, 1987); Traub, Wozniakowski, and Wasilkowski (1988); Motornyi, Ligun, and Doronin (1994); Zhensykbaev (2003).

Bojanov (1975) - one-dimensional analogue of problem 1 on classes W_p^r. A certain interpolating spline method is optimal.

Multidimensional results Babenko, Ligun (1975), Babenko (1978, 1980) for classes defined by a majorant of the modulus of continuity.

Proposition 1. *Let* $G \subset \mathbb{R}^d$, $d \in \mathbf{N}$, *be a convex body and* X *be a finite collection of points in* G. *Let*

$$S^*(f)(\mathbf{x}) = \frac{1}{2} \sup_{\substack{g \in W^2(G) \\ N(g)=N(f)}} g(\mathbf{x}) + \frac{1}{2} \inf_{\substack{g \in W^2(G) \\ N(g)=N(f)}} g(\mathbf{x}).$$

Then there holds

$$\inf_{\Phi} R(W^2(G); X, \Phi) = R(W^2(G); S^*)$$

Let

$$e(G, X_n) = \sup_{x \in G} \mathrm{dist}(\mathbf{x}, X_n).$$

Assume that $e(G, X) \geq \sqrt{2} \cdot e(\partial G, X)$. Then

$$R(W^2(G); S^*) = \frac{1}{4} e^2(G, X).$$

For $r > 0$, let

$$\psi_r(t) = \begin{cases} \frac{r^2}{4} - \frac{t^2}{2}, & t \in \left[0, \frac{r}{2}\right), \\ \frac{(t-r)^2}{2}, & t \in \left[\frac{r}{2}, r\right], \\ 0, & t \in (r, \infty). \end{cases}$$

Let $\mathbf{x}_0 \in G$ be the point such that

$$\mathrm{dist}(\mathrm{x}_0, X) = e(G, X).$$

Define $\Psi(\mathbf{x}) = \psi_r(|\mathbf{x} - \mathbf{x}_0|)$, where $r = e(G, X)$.

Optimal covering problem in $\mathbb{R}^d$: It is required to find the (infinite) collection of balls of radius 1, which covers $\mathbb{R}^d$ and has the least density.

Let Δ_d be the density of the most economical covering of $\mathbb{R}^d$.

Kershner (1939), Fejes-Toth (1940): $\Delta_2 = \frac{2\pi}{\sqrt{27}}$.

In $\mathbb{R}^d$, for certain $d \geq 3$, the most economical covering by balls with centers in a lattice is known.

If $G \subset \mathbb{R}^d$ is a convex body, it is known that

$$\inf_{\substack{\#X=n \\ X\subset G}} e(G,X) = \frac{\theta_d\, |G|^{1/d}}{N^{1/d}}(1+o(1)), \quad N \to \infty,$$

where

$$\theta_d = \left(\frac{\Delta_d}{|B[0,1]|}\right)^{1/d}.$$

For more references see e.g. books by Fejes-Toth (1953, 1964), Rogers (1964), Conway, Sloane (1999), Böröczky (2004).

Denote by U_d the collection of centers of the balls in the most economical covering of $\mathbb{R}^d$. Let $\mathbf{a}$ be some point inside G and

$$Y_h = (\mathbf{a} + hU_d) \cap G, \quad h > 0.$$

Denote by D_h the h-neighborhood of ∂G.

Let Z_h be a point set on $D_h \cap G$ such that

$$e(D_h \cap G, Z_h) \leq \frac{h}{\sqrt{2}}, \quad \text{and} \quad \#Z_h = o(\#Y_h), \quad h \to 0.$$

(One can take as Z_h an $h/\sqrt{2}$-separated subset of $D_h \cap G$ having the largest possible number of points.). Denote

$$n_h = \#(Y_h \cup Z_h), \quad h > 0. \tag{2}$$

Let

$$E_n(G) = \inf_{\substack{X \subset G \\ \#X = n}} e(G, X)$$

Theorem 1. *Let* $n, d \in \mathbf{N}$, $G \subset \mathbb{R}^d$ *be a convex body. There holds*

$$R_n(W^2(G)) = \frac{1}{4} E_n^2(G) \cdot (1 + o(1)) = \frac{\theta_d^2 \, |G|^{2/d}}{4N^{2/d}}(1 + o(1)), \quad N \to \infty.$$

The sequence of sets of nodes $Y_h \cup Z_h$, $h > 0$, *is asymptotically optimal on the class* $W^2(G)$, *i.e.*

$$R_{n_h}(W^2(G)) = R(W^2(G); Y_h \cup Z_h, \Phi_{n_h}^*)(1 + o(1)), \quad h \to 0. \quad (3)$$

Periodic case.

Let L be any full-rank lattice in $\mathbb{R}^d$, $d \in \mathbf{N}$, and C_L be the space of continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ such that for every $\mathbf{v} \in L$,

$$f(\mathbf{x}+\mathbf{v}) = f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d.$$

Denote by $W^2(L)$ the class of functions $f \in C_L \cap C^1(\mathbb{R}^d)$ such that

$$\left\| \frac{\partial^2 f}{\partial \mathbf{r}^2} \right\|_\infty := \operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}^d} \left| \frac{\partial^2 f}{\partial \mathbf{r}^2}(\mathbf{x}) \right| \le 1, \quad \text{for every} \quad \mathbf{r} \in \mathbb{R}^d, \quad |\mathbf{r}| = 1.$$

Denote by $\mathbf{v}_1, \ldots, \mathbf{v}_d$ the vectors from a basis of the lattice L and let

$$\Pi_d(L) = \{\alpha_1 \mathbf{v}_1 + \ldots + \alpha_d \mathbf{v}_d : \alpha_1, \ldots, \alpha_d \in [0,1)\}.$$

Let $X_n = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subset \Pi_d(L)$ and denote

$$N(f) = N(f, X_n) = (f(\mathbf{x}_1), \ldots, f(\mathbf{x}_n), \mathrm{grad} f(\mathbf{x}_1), \ldots, \mathrm{grad} f(\mathbf{x}_n)).$$

Let $\Phi_n : \mathbb{R}^{n(d+1)} \to C_L$ be an arbitrary mapping. We consider algorithms

$$S(f) = S(f; X_n, \Phi_n) = \Phi_n(N(f, X_n)). \tag{4}$$

Recovery error on the class $W^2(L)$:

$$R(W^2(L); S) = R(W^2(L); X_n, \Phi_n) = \sup_{f \in W^2(L)} \|f - S(f)\|_{\mathbb{R}^d}.$$

Problem 2. It is required to find the value

$$R_n(W^2(L)) = \inf_{\substack{X_n \subset \Pi_d(L) \\ \# X_n = n}} \inf_{\Phi_n} R(W^2(G); X_n, \Phi_n)$$

and optimal algorithms $S(f; X_n^*, \Phi_n^*)$, if they exist.

Proposition 2. Let L be a full-rank lattice in $\mathbb{R}^d$ and $X \subset \Pi_d(L)$ be a finite set such that

$$e(\mathbb{R}^d, X+L) < \frac{1}{2} \min_{\mathbf{v} \in L \setminus \{\mathbf{0}\}} |\mathbf{v}|.$$

Denote

$$S^*(f)(\mathbf{x}) = \frac{1}{2} \sup_{\substack{g \in W^2(L) \\ N(g)=N(f)}} g(\mathbf{x}) + \frac{1}{2} \inf_{\substack{g \in W^2(L) \\ N(g)=N(f)}} g(\mathbf{x}).$$

Then there holds

$$R(W^2(L); S^*) = \inf_{\Phi} R(W^2(L); X, \Phi) = \frac{1}{4} e^2(\mathbb{R}^d, X+L).$$

Let $X_n^* \subset \Pi_d(L)$ be such n-point set that

$$e(\mathbb{R}^d, X_n^* + L) = \inf_{\substack{X \subset \Pi_d(L) \\ \#X = n}} e(\mathbb{R}^d, X + L). \tag{5}$$

Theorem 2. Let $n, d \in \mathbf{N}$, L be a full-rank lattice in $\mathbb{R}^d$ and the relation

$$\inf_{\substack{X \subset G \\ \#X = n}} e(\mathbb{R}^d, X + L) < \frac{1}{2} \min_{\mathbf{v} \in L \setminus \{\mathbf{0}\}} |\mathbf{v}|.$$

The set of nodes X_n^*, defined above is the optimal set of nodes for the recovery of the class $W^2(L)$ by methods (4) with n nodes. In addition,

$$R_n(W^2(L)) = \frac{1}{4} e^2(\mathbb{R}^d, X_n^* + L) = \frac{\theta_d^2 \, |\Pi_d(L)|^{2/d}}{4N^{2/d}} (1 + o(1)), \quad N \to \infty.$$

Let L_0 be the lattice in $\mathbb{R}^2$ generated by vectors $\mathbf{v}_1 = (1,0)$ and $\mathbf{v}_2 = (0,\sqrt{3})$ and let L^* be the lattice in $\mathbb{R}^2$ generated by vectors $\mathbf{v}_1 = (1,0)$ and $\mathbf{v}_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Corollary 1. Let $n = 2N^2$, where N is a positive integer. The set of nodes

$$X_n^* = \left(\frac{1}{N}L^*\right) \cap \left([0,1) \times [0,\sqrt{3})\right)$$

is optimal on the class $W^2(L_0)$. In addition,

$$R_n(W^2(L_0)) = \frac{1}{6n}.$$

Corollary 2. Let $n = N^2$, where N is a positive integer. The set of nodes

$$X_n^* = \frac{1}{N} L^* \cap \Pi_2(L^*)$$

is optimal on the class $W^2(L^*)$. In addition,

$$R_n(W^2(L^*)) = \frac{1}{12n}.$$